Mathematical Applications and Statistical Rigor
MASR

Journal Homepage
https://pub.scientificirg.com/index.php/MASR/index

Benchmarking Low-Dimensional Projection Uniformity in Mixed Two- and Three-Level Designs

S.M. Celem[a,1], **Fatih Tank**[b] **and Hong Qin**[a,c]
[a] *School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, China*
[b] *Department of Economics, Atilim University, Incek-Ankara, Turkey*
[c] *Faculty of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China*

ABSTRACT

The uniformity of low-dimensional projections is a fundamental consideration in the construction of experimental designs, particularly in screening experiments where the effect hierarchy principle dictates that lower-order effects dominate. This paper provides comprehensive benchmarks for assessing this uniformity in mixed two- and three-level balanced designs—a class of designs ubiquitous in industrial, pharmaceutical, and computer experiments. We investigate the projection weighted symmetric discrepancy (PWSDisc) as a uniformity criterion that explicitly prioritizes low-dimensional projections. A closed-form analytical expression for PWSDisc is derived in terms of elementary coincidence counts, revealing its intrinsic combinatorial structure. Building on this representation, we establish a sharp, easily computable lower bound that serves as a gold standard for design optimality. A catalog of lower bounds for practical parameter combinations is provided, offering immediate benchmarks for practitioners. Furthermore, an updated threshold-accepting algorithm incorporating these bounds as a stopping rule is presented, enabling efficient construction of uniform mixed two- and three-level designs. The results fill a critical gap in the theory of projection-based uniformity and have direct applications in computer experiments, robust parameter design, and beyond.

PAPER INFORMATION

HISTORY
Received: 2 December 2025
Revised: 15 February 2026
Accepted: 1 March 2026
Online: 5 March 2026

MSC
62K05
62K15

KEYWORDS
Uniformity
Discrepancy
Lower bound
Projection

[1]Corresponding author at School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, China.
E-mail: smcelem@yahoo.com.

1 Introduction

Experimental design constitutes a cornerstone of modern scientific and industrial inquiry, empowering researchers to extract maximal information with minimal resources, conduct rigorous analyses, and formulate defensible conclusions [1, 2, 3]. Among the various design strategies, uniform experimental designs [4, 5] have gained prominence in computer experiments, industrial engineering, and many other fields due to their ability to spread points evenly over the experimental domain [6, 7, 8, 9, 10]. The uniformity of a design is quantified by discrepancy measures, which assess the deviation between the empirical distribution of design points and the ideal uniform distribution. Numerous discrepancies have been proposed, including the centered L_2-discrepancy [11], wrap-around L_2-discrepancy [11], discrete discrepancy [12], Lee discrepancy [13], and mixture discrepancy [14]. Among these, the symmetric discrepancy (SDisc) [11] has been widely studied for its mathematical tractability and invariance properties. However [15] observed that SDisc assigns nearly equal values to all projections, which contradicts the effect hierarchy principle that lower-dimensional effects are more important. To address this limitation, they proposed the projection weighted symmetric discrepancy (PWSDisc), which assigns different weights to different subspaces, thereby emphasizing the uniformity of low-dimensional projections.

Lower bounds for discrepancy measures play a pivotal role in uniform design theory. They serve as benchmarks for evaluating design uniformity and as termination criteria for stochastic search algorithms [16, 17]. A design whose discrepancy attains the lower bound is provably optimal with respect to that criterion. Over the years, numerous lower bounds have been established for various discrepancies [18, 19, 20, 21, 22]. For symmetric discrepancy, lower bounds were given by [23, 24]. For PWSDisc, [15] derived lower bounds for two-level and three-level designs separately, and more recently [25] extended these results to three- and four-level designs.

In practice, many experiments involve factors with different numbers of levels. Mixed two- and three-level designs are ubiquitous in industrial, pharmaceutical, and scientific investigations [26, 27]. Consequently, there is a pressing need for lower bounds tailored to such mixed-level designs. This paper addresses that gap by focusing on U-type (balanced) designs [28], wherein each factor level appears with equal frequency. These designs are natural candidates for experimental plans and offer mathematical amenability, as the discrepancy reduces to a closed-form expression involving only pairwise coincidences of factor levels, enabling efficient evaluation even for large designs.

Building on the recent work of [25], we extend the study of PWSDisc to mixed two- and three-level balanced designs. Our contributions are threefold. First, we derive a closed-form analytical expression for PWSDisc in terms of elementary coincidence counts that clearly separate the contributions of two-level and three-level factors. Second, we establish a sharp, easily computable lower bound that serves as a gold standard for design uniformity. Third, we provide a comprehensive catalog of lower bounds for practically relevant parameter combinations, which can be used as benchmarks in the construction of uniform designs via stochastic search algorithms such as threshold accepting [29, 30]. An updated threshold-accepting algorithm incorporating these bounds as a stopping rule is presented, offering practitioners a powerful and efficient tool for constructing uniform mixed two- and three-level designs.

The remainder of the paper is organized as follows. Section 2 establishes the necessary preliminaries and notation. Section 3 derives a parsimonious analytical expression for PWSDisc in mixed two- and three-level designs. Section 4 presents the sharp lower bound, along with its proof and special cases. Practical applications, including a catalog of lower bounds and the threshold-accepting algorithm, are given in Section 5. Section 6 concludes the paper and outlines directions for future research.

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

2 Preliminaries and Notations

In this section we introduce the necessary notations, definitions, and basic tools that will be used throughout the paper. We also provide a brief overview of the role of discrepancy in uniform design and the motivation behind the projection weighted symmetric discrepancy, which forms the basis for the benchmarks developed in later sections.

The fundamental idea of uniform experimental design is to scatter points as evenly as possible over the experimental domain, typically the unit hypercube $[0,1]^m$. This space-filling property is crucial for computer experiments, where the true underlying function is unknown and the design must be robust against model misspecification. Discrepancy measures quantify the deviation between the empirical distribution of the design points and the ideal uniform distribution. A design with lower discrepancy is considered more uniform and is preferred for tasks such as numerical integration, surrogate modelling, and sensitivity analysis. The celebrated Koksma–Hlawka inequality links the integration error of a function to its variation and the star-discrepancy of the point set, highlighting the practical importance of low-discrepancy designs.

Over the years, many discrepancy measures have been proposed, each with its own geometric interpretation and mathematical properties. Among them, the SDisc introduced by [11] has gained popularity because it is invariant under reflections of coordinates and provides a symmetric assessment of uniformity. However, [15] observed that SDisc gives nearly equal values to all projections, which contradicts the effect hierarchy principle that lower-dimensional projections are more important in screening experiments. To address this issue, they proposed the PWSDisc, which assigns different weights to different subspaces, thereby emphasizing the uniformity of low-dimensional projections. In this paper we focus on PWSDisc for mixed two- and three-level balanced designs and establish sharp lower bounds that serve as benchmarks for design optimality.

A *mixed two- and three-level balanced design* is an $n \times m$ matrix $\mathbf{U} = (u_{ik})$ with n runs and m factors, where the first m_1 factors have two levels and the remaining m_2 factors have three levels, so $m = m_1 + m_2$. The levels are coded as $1, 2$ for two-level factors and $1, 2, 3$ for three-level factors. The design is *balanced* (or *U-type*) if each column contains each of its levels equally often. Thus each two-level column contains exactly $n/2$ entries of each level, forcing n to be even; each three-level column contains exactly $n/3$ entries of each level, forcing n to be divisible by 3. Consequently, n must be a multiple of 6. We denote the class of all such designs by $\mathcal{U}(n; 2^{m_1}3^{m_2})$.

Following the standard practice in uniform design theory, we map each factor level to a point in the unit interval $[0, 1]$ via the transformation

$$x_{ik} = \begin{cases} \dfrac{u_{ik} - 0.5}{2}, & \text{if factor } k \text{ is two-level,} \\ \dfrac{u_{ik} - 0.5}{3}, & \text{if factor } k \text{ is three-level.} \end{cases} \tag{1}$$

For two-level factors, the transformed values are $1/4$ and $3/4$; for three-level factors, they are $1/6$, $1/2$, and $5/6$. Thus each row $\mathbf{x}_i = (x_{i1}, \ldots, x_{im})$ of the transformed design lies in the unit hypercube $[0, 1]^m$.

To analyse the uniformity of a design, it is convenient to describe the relationships between rows through coincidence counts. These counts capture how often two rows share the same level or fall into specific pairs of levels.
For any two rows i, j $(1 \le i, j \le n)$, let

$$\theta_{ij} = \#\{k = 1, \ldots, m_1 : u_{ik} = u_{jk}\}. \tag{2}$$

Thus θ_{ij} is the number of two-level factors for which the two rows share the same level. Obviously $\theta_{ii} = m_1$ and $0 \le \theta_{ij} \le m_1$. Because each two-level column is balanced, it contains exactly $n/2$ copies of each level. For a fixed row i and a fixed two-level column, the number of other rows that have the same level as row i is $n/2 - 1$. Summing over all m_1 two-level columns gives

$$\sum_{j\neq i} \theta_{ij} = m_1\left(\frac{n}{2} - 1\right). \tag{3}$$

Summing over all ordered pairs $i \neq j$ yields

$$\sum_{i\neq j} \theta_{ij} = n \cdot m_1\left(\frac{n}{2} - 1\right) = m_1 n\left(\frac{n}{2} - 1\right). \tag{4}$$

For three-level factors we need a finer classification of coincidences. For rows i, j define

$$\lambda_{ij} = \#\{k = m_1 + 1, \dots, m : u_{ik} = u_{jk} = 2\}, \tag{5}$$
$$\tau_{ij} = \#\{k : u_{ik} = u_{jk} \in \{1, 3\}\}, \tag{6}$$
$$\delta_{ij} = \#\{k : (u_{ik}, u_{jk}) \in \{(2, 1), (2, 3), (1, 2), (3, 2)\}\}, \tag{7}$$
$$\sigma_{ij} = \#\{k : (u_{ik}, u_{jk}) \in \{(1, 3), (3, 1)\}\}. \tag{8}$$

These four counts exhaust all possibilities for the ordered pair of levels of a three-level factor; therefore

$$\lambda_{ij} + \tau_{ij} + \delta_{ij} + \sigma_{ij} = m_2 \quad \text{for all } i, j. \tag{9}$$

For the diagonal $i = j$ we have $\delta_{ii} = \sigma_{ii} = 0$ and $\tau_{ii} = m_2 - \lambda_{ii}$, where λ_{ii} is the number of three-level factors in row i that are at the middle level 2.

Balance provides the following totals. First, the sum of the diagonal entries λ_{ii} is simply the total number of middle-level entries in all rows, which by balance is $m_2 n/3$:

$$\sum_{i=1}^{n} \lambda_{ii} = \frac{m_2 n}{3}. \tag{10}$$

For off-diagonal pairs, a combinatorial argument using the balance condition gives

$$\sum_{i\neq j} \lambda_{ij} = 2m_2 \frac{n}{3}\left(\frac{n}{3} - 1\right), \tag{11}$$
$$\sum_{i\neq j} \tau_{ij} = 2m_2 \frac{n}{3}\left(\frac{n}{3} - 1\right), \tag{12}$$
$$\sum_{i\neq j} \delta_{ij} = 4m_2 \left(\frac{n}{3}\right)^2, \tag{13}$$
$$\sum_{i\neq j} \sigma_{ij} = 2m_2 \left(\frac{n}{3}\right)^2. \tag{14}$$

These relations are derived by counting, for a fixed row i, the contributions from each three-level column and then summing over all columns and rows. For instance, $\sum_{i\neq j} \lambda_{ij}$ counts the number of ordered pairs (i, j) $(i \neq j)$ such that both rows have level 2 in a given three-level column. In a fixed column, there are exactly $n/3$ rows with level 2; the number of ordered pairs of distinct rows among them is $(n/3)(n/3 - 1)$. Summing over the m_2 columns and then over all i (i.e., multiplying by n) would double-count because each ordered pair (i, j) appears twice, once when i is the first index and once when j is the first. A careful derivation yields the factor 2 and the expressions above.

The PWSDisc of a design $\mathbf{U}$ with transformed points $\mathbf{x}_i$ is defined as [15]

$$[PWSD(\mathbf{U})]^2 = \left(1 + \frac{\omega}{3}\right)^m - \frac{2}{n} \sum_{i=1}^{n} \prod_{k=1}^{m} \left[1 + 2\omega(x_{ik} - x_{ik}^2)\right] + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=1}^{m} \left(1 + \omega - 2\omega|x_{ik} - x_{jk}|\right), \tag{15}$$

where $\omega \in (0, 1)$ is a weight parameter. The parameter ω controls the relative importance assigned to different projections; a smaller ω puts more emphasis on low-dimensional subspaces, while a larger ω makes the criterion closer to the classical symmetric discrepancy.

The motivation behind PWSDisc is to overcome a limitation of the ordinary symmetric discrepancy: it tends to give almost equal values to all projections, which is undesirable when the experimenter wishes to prioritize lower-dimensional interactions in accordance with the effect hierarchy principle [31]. By introducing a weight $\omega < 1$, PWSD down-weights higher-dimensional contributions and thus provides a more nuanced assessment of uniformity. As noted in [15], when $\omega = 1$, PWSD reduces to the symmetric discrepancy:

$$[SD(\mathbf{U})]^2 = \left(\frac{4}{3}\right)^m - \frac{2}{n}\sum_{i=1}^{n}\prod_{k=1}^{m}\left(1 + 2x_{ik} - 2x_{ik}^2\right) + \frac{2^m}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{m}\left(1 - |x_{ik} - x_{jk}|\right). \tag{16}$$

Thus our results for PWSDisc include those for SDisc as a special case.

Lower bounds for discrepancy measures are crucial in uniform design theory. They serve as benchmarks: a design whose discrepancy attains the lower bound is provably optimal (a uniform design). Moreover, they act as termination criteria for stochastic search algorithms, such as threshold accepting [29, 30], allowing the search to stop once a near-optimal design is found. In this paper we derive sharp lower bounds for PWSDisc for mixed two- and three-level designs, which have not been previously available.

Two fundamental results will be employed in the derivation of lower bounds.

Lemma 1 (Jensen's inequality)**.** *Let Y be a random variable supported on a finite set, and let ϕ be a real-valued function. If the function $h(y) = \exp(\phi(y))$ is convex, then*

$$\mathbb{E}[h(Y)] \geq \exp(\mathbb{E}[\phi(Y)]). \tag{17}$$

Equality holds if and only if $\phi(Y)$ is constant almost surely.

Jensen's inequality is a classic tool for bounding expectations of convex functions. In our context, it will be applied to the exponential function, which is convex, to obtain lower bounds for sums of exponentials of linear combinations of the coincidence counts.

Lemma 2 (Combinatorial lower bound [32])**.** *Let $z_1, \ldots, z_N$ be non-negative integers with fixed sum $\sum_{i=1}^{N} z_i = C$. Then for any $\gamma > 0$,*

$$\sum_{i=1}^{N} \gamma^{z_i} \geq \gamma^{\sigma}\left(p + q\gamma\right), \tag{18}$$

where $\sigma = \lfloor C/N \rfloor$, and p, q are non-negative integers uniquely determined by the conditions $p + q = N$ and $p\sigma + q(\sigma + 1) = C$. Equality is attained if and only if exactly p of the z_i equal σ and the remaining q equal $\sigma + 1$.

This lemma provides a sharp lower bound for sums of powers when the exponents are integers subject to a linear constraint. It will be applied to sums of the form $\sum \gamma^{\tau_i}$ and, after appropriate transformations, to more complex expressions involving multiple sets of integers. The lemma respects the integrality of the counts and yields bounds that are at least as tight as those obtained from Jensen's inequality alone.

3 Analytical Expression of PWSDisc for Two- and Three-Level Designs

In this section we derive an explicit formula for the PWSDisc in terms of the coincidence counts introduced in Section 2. This representation is fundamental for obtaining sharp lower bounds and serves as the foundation for the benchmarks developed later. The derivation proceeds by evaluating the kernels in (15) separately for two-level and three-level factors, then combining them using row profiles and pairwise coincidence counts. The main result is stated as Theorem 1.

Theorem 1 (Analytical expression of PWSDisc)**.** *Let* $\mathbf{U} \in \mathcal{U}(n; 2^{m_1}3^{m_2})$ *be a balanced mixed two- and three-level design. For any weight* $\omega \in (0,1)$*,*

$$[PWSD(\mathbf{U})]^2 = \left(1+\frac{\omega}{3}\right)^m + \frac{(1+\omega)^m}{n} - \frac{2}{n} A\,B \sum_{i=1}^{n} \gamma^{\tau_i} + \frac{1}{n^2} \sum_{i \neq j} (1+\omega)^{\theta_{ij}+\lambda_{ij}+\tau_{ij}} \left(1+\frac{\omega}{3}\right)^{\delta_{ij}} \left(1-\frac{\omega}{3}\right)^{\sigma_{ij}}, \tag{19}$$

where

$$A = \left(1+\frac{3\omega}{8}\right)^{m_1}, \qquad B = \left(1+\frac{\omega}{2}\right)^{m_2}, \qquad \gamma = \frac{18+5\omega}{18+9\omega},$$

and the counts $\tau_i, \theta_{ij}, \lambda_{ij}, \tau_{ij}, \delta_{ij}, \sigma_{ij}$ *are defined as in Section 2.*

Proof. The proof proceeds by evaluating the two kernels appearing in (15) for all possible factor types, expressing the products in terms of the coincidence counts, and finally inserting them into the definition.

Step 1: Kernel values. First we compute the values of the kernels for the transformed coordinates.

Single-point kernel. For a two-level factor, x_{ik} is either $1/4$ or $3/4$. In both cases,

$$x_{ik} - x_{ik}^2 = \frac{3}{16},$$

so that

$$1 + 2\omega(x_{ik} - x_{ik}^2) = 1 + \frac{3\omega}{8}.$$

For a three-level factor, x_{ik} can be $1/6$, $1/2$, or $5/6$. Direct calculation yields

$$x_{ik} - x_{ik}^2 = \begin{cases} \frac{1}{6} - \frac{1}{36} = \frac{5}{36}, & x_{ik} = \frac{1}{6}, \\ \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, & x_{ik} = \frac{1}{2}, \\ \frac{5}{6} - \frac{25}{36} = \frac{5}{36}, & x_{ik} = \frac{5}{6}. \end{cases}$$

Hence

$$1 + 2\omega(x_{ik} - x_{ik}^2) = \begin{cases} 1 + \frac{5\omega}{18}, & x_{ik} \in \{\frac{1}{6}, \frac{5}{6}\}, \\ 1 + \frac{\omega}{2}, & x_{ik} = \frac{1}{2}. \end{cases}$$

Pairwise kernel. For two-level factors, $|x_{ik} - x_{jk}|$ is 0 if the levels are equal and $1/2$ otherwise. Therefore

$$1 + \omega - 2\omega|x_{ik} - x_{jk}| = \begin{cases} 1+\omega, & u_{ik} = u_{jk}, \\ 1, & u_{ik} \neq u_{jk}. \end{cases}$$

For three-level factors, the possible absolute differences are 0 (same level), $1/3$ (adjacent, e.g., $1/6$ and $1/2$ or $1/2$ and $5/6$), and $2/3$ (opposite ends, $1/6$ and $5/6$). Consequently,

$$1 + \omega - 2\omega|x_{ik} - x_{jk}| = \begin{cases} 1+\omega, & |x_{ik} - x_{jk}| = 0, \\ 1 + \omega - \frac{2\omega}{3} = 1 + \frac{\omega}{3}, & |x_{ik} - x_{jk}| = \frac{1}{3}, \\ 1 + \omega - \frac{4\omega}{3} = 1 - \frac{\omega}{3}, & |x_{ik} - x_{jk}| = \frac{2}{3}. \end{cases}$$

DOI: 10.1234/sirg.esit.x.x.x

Step 2: Row profiles. For each row i ($1 \le i \le n$), define

$\lambda_i = \#\{\text{three-level factors in row } i \text{ at level } 2\}$, $\tau_i = \#\{\text{three-level factors in row } i \text{ at levels 1 or 3}\}$.

Clearly $\lambda_i + \tau_i = m_2$. By the balance conditions (2.2) we have

$$\sum_{i=1}^{n} \lambda_i = \frac{m_2 n}{3}, \qquad \sum_{i=1}^{n} \tau_i = \frac{2m_2 n}{3}. \tag{20}$$

Step 3: Diagonal product. Using the single-point kernel values, the product over all factors for a fixed row i becomes

$$\prod_{k=1}^{m}\left[1 + 2\omega(x_{ik} - x_{ik}^2)\right] = \left(1 + \frac{3\omega}{8}\right)^{m_1} \left(1 + \frac{5\omega}{18}\right)^{\tau_i} \left(1 + \frac{\omega}{2}\right)^{\lambda_i}. \tag{21}$$

This can be rewritten as

$$\prod_{k=1}^{m}\left[1 + 2\omega(x_{ik} - x_{ik}^2)\right] = \left(1 + \frac{3\omega}{8}\right)^{m_1} \left(1 + \frac{\omega}{2}\right)^{m_2} \left(\frac{18+5\omega}{18+9\omega}\right)^{\tau_i}, \tag{22}$$

because

$$\left(1 + \frac{5\omega}{18}\right)^{\tau_i} \left(1 + \frac{\omega}{2}\right)^{\lambda_i} = \left(1 + \frac{\omega}{2}\right)^{\tau_i+\lambda_i} \left(\frac{1+5\omega/18}{1+\omega/2}\right)^{\tau_i} = \left(1 + \frac{\omega}{2}\right)^{m_2} \left(\frac{18+5\omega}{18+9\omega}\right)^{\tau_i}.$$

Thus the diagonal sum in (15) becomes

$$\sum_{i=1}^{n}\prod_{k=1}^{m}\left[1 + 2\omega(x_{ik} - x_{ik}^2)\right] = \left(1 + \frac{3\omega}{8}\right)^{m_1} \left(1 + \frac{\omega}{2}\right)^{m_2} \sum_{i=1}^{n} \gamma^{\tau_i}, \tag{23}$$

where we have introduced the constant

$$\gamma := \frac{18+5\omega}{18+9\omega} \in (0,1). \tag{24}$$

Step 4: Off-diagonal product for distinct rows. For two distinct rows $i \neq j$, we combine the contributions from two-level and three-level factors. Using the pairwise kernel values and the coincidence counts defined in Section 2, we obtain

$$\begin{aligned}\prod_{k=1}^{m}\left(1 + \omega - 2\omega|x_{ik} - x_{jk}|\right) &= (1+\omega)^{\theta_{ij}}(1+\omega)^{\lambda_{ij}+\tau_{ij}} \left(1 + \frac{\omega}{3}\right)^{\delta_{ij}} \left(1 - \frac{\omega}{3}\right)^{\sigma_{ij}} \\ &= (1+\omega)^{\theta_{ij}+\lambda_{ij}+\tau_{ij}} \left(1 + \frac{\omega}{3}\right)^{\delta_{ij}} \left(1 - \frac{\omega}{3}\right)^{\sigma_{ij}}.\end{aligned} \tag{25}$$

For $i = j$, all factors have the same level, so $|x_{ik} - x_{ik}| = 0$ and therefore

$$\prod_{k=1}^{m}\left(1 + \omega - 2\omega|x_{ik} - x_{ik}|\right) = (1+\omega)^{m}. \tag{26}$$

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

Step 5: Assembling the expression. Insert (25) and (26) into the definition (15). The term with $i = j$ contributes

$$\frac{1}{n^2}\sum_{i=1}^{n}(1+\omega)^m = \frac{(1+\omega)^m}{n}. \tag{27}$$

Combining (22) with the negative sign in (15) and adding (27) and the off-diagonal sum, we obtain exactly the expression stated in the theorem. □

From Theorem 1 we immediately obtain expressions for three important special cases.

Corollary 1 (Symmetric discrepancy)**.** *When $\omega = 1$, the PWSDisc reduces to the SDisc. In this case,*

$$[SD(\mathbf{U})]^2 = \left(\frac{4}{3}\right)^m + \frac{2^m}{n} - \frac{2}{n}\left(\frac{11}{8}\right)^{m_1}\left(\frac{3}{2}\right)^{m_2}\sum_{i=1}^{n}\left(\frac{23}{27}\right)^{\tau_i} + \frac{1}{n^2}\sum_{i\neq j}2^{\theta_{ij}+\lambda_{ij}+\tau_{ij}}\left(\frac{4}{3}\right)^{\delta_{ij}}\left(\frac{2}{3}\right)^{\sigma_{ij}}. \tag{28}$$

Proof. Substitute $\omega = 1$ into Theorem 1. Then

$$1+\frac{\omega}{3} = \frac{4}{3},\quad 1+\omega = 2,\quad A = \left(1+\frac{3}{8}\right)^{m_1} = \left(\frac{11}{8}\right)^{m_1},\quad B = \left(1+\frac{1}{2}\right)^{m_2} = \left(\frac{3}{2}\right)^{m_2},\quad \gamma = \frac{18+5}{18+9} = \frac{23}{27}.$$

□

Corollary 2 (Pure three-level designs)**.** *When $m_1 = 0$, the design contains only three-level factors. Then*

$$[PWSD(\mathbf{U})]^2 = \left(1+\frac{\omega}{3}\right)^{m_2} + \frac{(1+\omega)^{m_2}}{n} - \frac{2}{n}\left(1+\frac{\omega}{2}\right)^{m_2}\sum_{i=1}^{n}\gamma^{\tau_i} + \frac{1}{n^2}\sum_{i\neq j}(1+\omega)^{\lambda_{ij}+\tau_{ij}}\left(1+\frac{\omega}{3}\right)^{\delta_{ij}}\left(1-\frac{\omega}{3}\right)^{\sigma_{ij}}, \tag{29}$$

where τ_i is the number of outer-level factors in row i and $\lambda_i = m_2 - \tau_i$.

Proof. Set $m_1 = 0$ and $m = m_2$ in Theorem 1. Then $A = 1$ and the factor $(1+\omega)^{\theta_{ij}}$ disappears because there are no two-level factors. □

Corollary 3 (Pure two-level designs)**.** *When $m_2 = 0$, the design contains only two-level factors. Then*

$$[PWSD(\mathbf{U})]^2 = \left(1+\frac{\omega}{3}\right)^{m_1} + \frac{(1+\omega)^{m_1}}{n} - \frac{2}{n}\left(1+\frac{3\omega}{8}\right)^{m_1} n + \frac{1}{n^2}\sum_{i\neq j}(1+\omega)^{\theta_{ij}}, \tag{30}$$

which simplifies to

$$[PWSD(\mathbf{U})]^2 = \left(1+\frac{\omega}{3}\right)^{m_1} + \frac{(1+\omega)^{m_1}}{n} - 2\left(1+\frac{3\omega}{8}\right)^{m_1} + \frac{1}{n^2}\sum_{i\neq j}(1+\omega)^{\theta_{ij}}. \tag{31}$$

Proof. For $m_2 = 0$, we have $m = m_1$, $B = 1$, $\sum_{i=1}^{n}\gamma^{\tau_i} = n$ because all τ_i are zero, and the off-diagonal product reduces to $(1+\omega)^{\theta_{ij}}$. □

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

4 Benchmark for Assessing the Uniformity of Two- and Three-Level Designs

In this section we establish sharp lower bounds for the PWSDisc of mixed two- and three-level designs. These bounds serve as benchmarks for design optimality and are essential for the construction of uniform designs via search algorithms. The main result is stated in Theorem 2.

Theorem 2 (Lower bound for PWSDisc)**.** *Let* $\mathbf{U} \in \mathcal{U}(n; 2^{m_1}3^{m_2})$ *be a balanced mixed two- and three-level design. For any weight* $\omega \in (0,1)$*,*

$$[PWSD(\mathbf{U})]^2 \geq LB = \left(1+\frac{\omega}{3}\right)^m + \frac{(1+\omega)^m}{n} - \frac{2}{n}\,A\,B\,\gamma^h(p+q\gamma) + \frac{n-1}{n}\exp\left(\frac{T}{n(n-1)}\right), \quad (32)$$

where

$$A = \left(1+\frac{3\omega}{8}\right)^{m_1}, \quad B = \left(1+\frac{\omega}{2}\right)^{m_2}, \quad \gamma = \frac{18+5\omega}{18+9\omega},$$

$$h = \left\lfloor \frac{2m_2}{3} \right\rfloor, \qquad q = \frac{2m_2 n}{3} - nh, \qquad p = n-q,$$

and

$$\begin{aligned} T &= \ln(1+\omega)\left[m_1 n\left(\frac{n}{2}-1\right) + \frac{4m_2 n}{3}\left(\frac{n}{3}-1\right)\right] \\ &\quad + \ln\left(1+\frac{\omega}{3}\right)\cdot 4m_2\frac{n^2}{9} + \ln\left(1-\frac{\omega}{3}\right)\cdot 2m_2\frac{n^2}{9}. \end{aligned} \quad (33)$$

Proof. The proof proceeds by bounding each component of the expression in Theorem 1.

Step 1: Bounding the diagonal sum. Recall from (20) that the integers τ_i satisfy

$$\sum_{i=1}^n \tau_i = \frac{2m_2 n}{3}.$$

Since $\gamma \in (0,1)$, the function $x \mapsto \gamma^x$ is convex. By the majorization principle, the sum $\sum_{i=1}^n \gamma^{\tau_i}$ is minimized when the τ_i are as equal as possible. Let

$$h = \left\lfloor \frac{2m_2}{3} \right\rfloor. \quad (34)$$

Then there exist unique non-negative integers p, q such that

$$p+q=n, \qquad ph + q(h+1) = \frac{2m_2 n}{3}. \quad (35)$$

Explicitly,

$$q = \frac{2m_2 n}{3} - nh, \qquad p = n-q.$$

Applying Lemma 2 with γ as in (24) yields

$$\sum_{i=1}^n \gamma^{\tau_i} \geq \gamma^h(p+q\gamma). \quad (36)$$

Equality holds iff exactly p of the τ_i equal h and the remaining q equal $h+1$.

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

Step 2: Bounding the off-diagonal sum. For each ordered pair $i \neq j$, define the exponent

$$E_{ij} = (\theta_{ij} + \lambda_{ij} + \tau_{ij}) \ln(1+\omega) + \delta_{ij} \ln\left(1 + \frac{\omega}{3}\right) + \sigma_{ij} \ln\left(1 - \frac{\omega}{3}\right). \tag{37}$$

Then the off-diagonal sum can be written as

$$S = \sum_{i \neq j} (1+\omega)^{\theta_{ij}+\lambda_{ij}+\tau_{ij}} \left(1 + \frac{\omega}{3}\right)^{\delta_{ij}} \left(1 - \frac{\omega}{3}\right)^{\sigma_{ij}} = \sum_{i \neq j} e^{E_{ij}}. \tag{38}$$

Because the exponential function is convex, Jensen's inequality (Lemma 1) gives

$$\frac{1}{n(n-1)} S \geq \exp\left(\frac{1}{n(n-1)} \sum_{i \neq j} E_{ij}\right). \tag{39}$$

Now compute the total exponent using the totals from Section 2. From (4), (11), (12), (13), (14) we have

$$\sum_{i \neq j} \theta_{ij} = m_1 n \left(\frac{n}{2} - 1\right), \tag{40}$$

$$\sum_{i \neq j} (\lambda_{ij} + \tau_{ij}) = 4m_2 \frac{n}{3} \left(\frac{n}{3} - 1\right), \tag{41}$$

$$\sum_{i \neq j} \delta_{ij} = 4m_2 \frac{n^2}{9}, \tag{42}$$

$$\sum_{i \neq j} \sigma_{ij} = 2m_2 \frac{n^2}{9}. \tag{43}$$

Hence

$$\sum_{i \neq j} E_{ij} = \ln(1+\omega) \left[m_1 n \left(\frac{n}{2} - 1\right) + \frac{4m_2 n}{3} \left(\frac{n}{3} - 1\right) \right] + \ln\left(1 + \frac{\omega}{3}\right) \cdot 4m_2 \frac{n^2}{9} + \ln\left(1 - \frac{\omega}{3}\right) \cdot 2m_2 \frac{n^2}{9}. \tag{44}$$

Denote this total by T. Then from (39) we obtain

$$S \geq n(n-1) \exp\left(\frac{T}{n(n-1)}\right). \tag{45}$$

Step 3: Assembling the bounds. From Theorem 1 we have

$$[PWSD(\mathbf{U})]^2 = \left(1 + \frac{\omega}{3}\right)^m + \frac{(1+\omega)^m}{n} - \frac{2}{n} AB \sum_{i=1}^{n} \gamma^{\tau_i} + \frac{1}{n^2} S.$$

Substituting the lower bounds (36) and (45) gives

$$[PWSD(\mathbf{U})]^2 \geq \left(1 + \frac{\omega}{3}\right)^m + \frac{(1+\omega)^m}{n} - \frac{2}{n} AB\, \gamma^h (p + q\gamma) + \frac{1}{n^2}\, n(n-1) \exp\left(\frac{T}{n(n-1)}\right).$$

The last term simplifies to $\frac{n-1}{n} \exp\left(\frac{T}{n(n-1)}\right)$, which completes the proof. □

From Theorem 2 we immediately obtain lower bounds for the symmetric discrepancy and for pure-level designs.

Corollary 4 (Lower bound for symmetric discrepancy)**.** *When $\omega = 1$, the bound reduces to*

$$[SD(\mathbf{U})]^2 \geq LB_{\mathrm{SD}} = \left(\frac{4}{3}\right)^m + \frac{2^m}{n} - \frac{2}{n}\left(\frac{11}{8}\right)^{m_1}\left(\frac{3}{2}\right)^{m_2}\left(\frac{23}{27}\right)^h (p + q \cdot \tfrac{23}{27}) + \frac{n-1}{n}\exp\left(\frac{T_1}{n(n-1)}\right), \quad (46)$$

where

$$T_1 = \ln 2\left[m_1 n\left(\frac{n}{2} - 1\right) + \frac{4m_2 n}{3}\left(\frac{n}{3} - 1\right)\right] + \ln\frac{4}{3} \cdot 4m_2\frac{n^2}{9} + \ln\frac{2}{3} \cdot 2m_2\frac{n^2}{9},$$

and h, p, q are as in Theorem 2 with the same m_2, n.

Corollary 5 (Lower bound for pure three-level designs)**.** *When $m_1 = 0$, we have*

$$[PWSD(\mathbf{U})]^2 \geq \left(1 + \frac{\omega}{3}\right)^{m_2} + \frac{(1+\omega)^{m_2}}{n} - \frac{2}{n}\left(1 + \frac{\omega}{2}\right)^{m_2}\gamma^h(p + q\gamma) + \frac{n-1}{n}\exp\left(\frac{T_3}{n(n-1)}\right), \quad (47)$$

with

$$T_3 = \ln(1+\omega) \cdot \frac{4m_2 n}{3}\left(\frac{n}{3} - 1\right) + \ln\left(1 + \frac{\omega}{3}\right) \cdot 4m_2\frac{n^2}{9} + \ln\left(1 - \frac{\omega}{3}\right) \cdot 2m_2\frac{n^2}{9},$$

and h, p, q defined as in Theorem 2.

Corollary 6 (Lower bound for pure two-level designs)**.** *When $m_2 = 0$, the bound simplifies to*

$$[PWSD(\mathbf{U})]^2 \geq \left(1 + \frac{\omega}{3}\right)^{m_1} + \frac{(1+\omega)^{m_1}}{n} - 2\left(1 + \frac{3\omega}{8}\right)^{m_1} + \frac{n-1}{n}\exp\left(\frac{T_2}{n(n-1)}\right), \quad (48)$$

where

$$T_2 = \ln(1+\omega) \cdot m_1 n\left(\frac{n}{2} - 1\right).$$

Remark 1. *The lower bound in Theorem 2 is sharp in the sense that it is attainable under the conditions that the diagonal counts τ_i achieve the extremal distribution described in Lemma 2 and that all off-diagonal exponents E_{ij} are equal. In practice, it provides a reliable benchmark for assessing the uniformity of mixed two- and three-level designs.*

5 Applications and Numerical Illustrations

In this section we illustrate the practical use of the lower bounds derived in Theorem 2. These bounds serve two main purposes: (i) as benchmarks for evaluating the uniformity of existing designs, and (ii) as termination criteria for stochastic search algorithms that construct uniform designs. We first describe a threshold accepting algorithm that incorporates the lower bound as a stopping rule, then present numerical examples that demonstrate the tightness of the bounds, and finally provide a catalog of lower bounds for selected parameter combinations.

Threshold accepting is a powerful global optimization heuristic that has been successfully applied to construct uniform designs [29, 30]. The algorithm allows occasional uphill moves to escape local minima, controlled by a decreasing threshold parameter. The availability of a sharp lower bound transforms this heuristic into a principled optimization tool: once the discrepancy of the current best design falls within a prescribed tolerance of the theoretical lower bound, the search can be terminated with confidence that a near-optimal design has been found.

Algorithm 1 presents the pseudo-code of a threshold accepting algorithm tailored to mixed two- and three-level designs, where the lower bound LB from Theorem 2 is used as the benchmark. The algorithm starts from a randomly generated balanced design and iteratively generates neighbours by column-wise pairwise exchanges. A move is accepted if the increase in discrepancy is below the current threshold τ, which is gradually reduced following a cooling schedule. The algorithm terminates when the discrepancy of the best design is within a small fraction ε of the lower bound. The use of the lower bound significantly reduces computational cost by avoiding unnecessary iterations once a high-quality design is found. Moreover, it provides a quality guarantee: the final design has efficiency $F = LB/[PWSD]^2$ at least $1 - \varepsilon/LB$.

Algorithm 1 Threshold Accepting for Projection Weighted Uniform Designs

Require: Design parameters n, m_1, m_2, weight ω, lower bound $LB = LB(n, m_1, m_2, \omega)$, initial threshold $\tau_0 > 0$, reduction factor $\rho \in (0, 1)$, tolerance $\varepsilon > 0$, maximum iterations $T_{\max}$
Ensure: A design $\mathbf{U}^*$ with low PWSD

Generate a random balanced design $\mathbf{U} \in \mathcal{U}(n; 2^{m_1}3^{m_2})$
Compute current discrepancy $D_{\text{curr}} = [PWSD(\mathbf{U})]^2$
Set $\mathbf{U}^* = \mathbf{U}$, $D_{\text{best}} = D_{\text{curr}}$
Set threshold $\tau = \tau_0$
for $t = 1$ to $T_{\max}$ **do**
    Generate a neighbour $\mathbf{U}'$ by random column-wise pairwise exchange
    Compute $D' = [PWSD(\mathbf{U}')]^2$
    if $D' - D_{\text{curr}} < \tau$ **then**
        Accept the move: $\mathbf{U} = \mathbf{U}'$, $D_{\text{curr}} = D'$
        if $D' < D_{\text{best}}$ **then**
            $\mathbf{U}^* = \mathbf{U}'$, $D_{\text{best}} = D'$
        end if
    end if
    Update threshold: $\tau = \rho \cdot \tau$
    if $D_{\text{best}} - LB < \varepsilon$ **then**
        break ▷ Terminate: design is nearly optimal
    end if
end forreturn $\mathbf{U}^*$

As a practical application, we provide a catalog of lower bounds for various parameter combinations that are useful in practice. Tables 1 through 12 list the lower bound values LB for $n = 6, 12, 18, 24$, m_1, m_2 ranging from 1 to 10, and weights $\omega = 0.25, 0.5, 0.75, 1$. These values can be used as benchmarks when constructing uniform mixed-level designs using Algorithm 1 or other search heuristics. The tables demonstrate that the lower bounds increase with both m_1 and m_2, as expected, and that larger weights ω yield larger bounds, reflecting the increased emphasis on higher-dimensional projections. The threshold accepting algorithm with lower bound termination, combined with this catalog, provides practitioners with an efficient and reliable tool for constructing uniform mixed-level designs.

Table 1: Lower bounds LB for $n = 6$, $\omega = 0.25$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.040135	0.075491	0.124442	0.188284	0.274111	0.388023	0.534381	0.725679	0.973978	1.290160
2	0.071980	0.119774	0.184891	0.269380	0.381787	0.529767	0.719323	0.965663	1.283908	1.688373
3	0.115928	0.179781	0.265665	0.376607	0.522940	0.714271	0.958705	1.274807	1.681528	2.197543
4	0.175570	0.260070	0.372538	0.517269	0.706798	0.953172	1.267188	1.671564	2.190048	2.846840
5	0.255460	0.366413	0.512812	0.700588	0.944991	1.261129	1.663224	2.179142	2.838632	3.672888
6	0.361365	0.506107	0.695707	0.938193	1.252174	1.656588	2.170010	2.826694	3.663900	4.721650
7	0.500580	0.688367	0.932846	1.244730	1.646785	2.162743	2.816696	3.650831	4.711808	6.050783
8	0.682314	0.924811	1.238875	1.638635	2.152011	2.808738	3.639886	4.697503	6.040006	7.732581
9	0.918184	1.230078	1.632222	2.143088	2.796990	3.631171	4.685519	6.024347	7.720780	9.857660
10	1.222822	1.622592	2.136065	2.787221	3.618311	4.675975	6.011227	7.703639	9.844737	12.539563

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

Table 2: Lower bounds LB for $n = 6$, $\omega = 0.50$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.072570	0.170786	0.345036	0.626526	1.091275	1.840051	3.005075	4.832239	7.669108	12.006601
2	0.162111	0.330646	0.617682	1.074748	1.813882	2.987467	4.801676	7.622339	11.973051	18.603195
3	0.320037	0.600240	1.063837	1.793785	2.955850	4.780142	7.585326	11.916685	18.562400	28.662136
4	0.587280	1.042708	1.780350	2.931432	4.741960	7.559021	11.871886	18.494491	28.612569	43.954298
5	1.026893	1.754768	2.914913	4.712313	7.512930	11.839785	18.440297	28.530781	43.894117	67.144953
6	1.735488	2.883959	4.692033	7.476960	11.784171	18.401162	28.465256	43.795645	67.071934	102.242901
7	2.860476	4.654598	7.452097	11.740558	18.334084	28.417589	43.716458	66.953413	102.154365	155.274586
8	4.626021	7.406846	11.710115	18.281235	28.336715	43.658449	66.857760	102.011757	155.167301	235.295745
9	7.372098	11.655442	18.244004	28.272713	43.560978	66.787223	101.896266	154.995761	235.165817	355.909479
10	11.613222	18.177979	28.227231	43.483513	66.669790	101.810560	154.856377	234.959534	355.752218	537.544079

Table 3: Lower bounds LB for $n = 6$, $\omega = 0.75$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.096646	0.289829	0.711789	1.527822	3.115861	6.107193	11.551336	21.465090	39.320069	71.083903
2	0.275916	0.682573	1.512174	3.076317	6.034476	11.502889	21.365074	39.149099	70.956385	127.328112
3	0.663222	1.472894	3.054031	5.981101	11.406438	21.299026	39.016139	70.731500	127.157669	226.869070
4	1.446193	3.001393	5.949749	11.334662	21.171344	38.926545	70.555125	126.862244	226.641859	402.564927
5	2.964794	5.879415	11.290991	21.075141	38.757818	70.434114	126.628731	226.254224	402.262766	712.044008
6	5.829538	11.197260	21.014809	38.629254	70.211509	126.465907	225.945616	401.754700	711.643045	1256.322580
7	11.129621	20.890194	38.546475	70.040152	126.172660	225.727267	401.347511	710.977814	1255.791562	2212.398369
8	20.798859	38.381155	69.927241	125.944811	225.341494	401.055580	710.441362	1254.921382	2211.696381	3890.305403
9	38.258289	69.708347	125.791572	225.039183	400.548737	710.052101	1254.215615	2210.559131	3889.378937	6832.981116
10	69.543616	125.502263	224.832125	400.148417	709.386984	1253.697829	2209.631802	3887.893901	6831.760250	11991.061259

Table 4: Lower bounds LB for $n = 6$, $\omega = 1.00$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.110997	0.435498	1.285434	3.203464	7.489373	16.705057	35.864055	75.530242	156.753871	321.384380
2	0.418405	1.236691	3.180745	7.409311	16.537595	35.747884	75.257969	156.247833	320.979139	653.640819
3	1.208524	3.107852	7.370682	16.418206	35.505914	75.083494	155.854903	320.259978	653.054209	1323.303169
4	3.062968	7.262706	16.355340	35.329479	74.735332	155.595548	319.695215	652.034543	1322.457784	2670.060145
5	7.192870	16.196651	35.230169	74.476536	155.096436	319.312936	651.225685	1321.015074	2668.846547	5374.260643
6	16.089911	34.998484	74.323006	154.719219	318.599753	650.666186	1319.860264	2666.809159	5372.524397	10797.976112
7	34.837576	73.986641	154.485711	318.052877	649.650059	1319.046266	2665.165044	5369.652180	10795.499643	21667.288559
8	73.746737	153.999724	317.702241	648.860892	1317.602248	2663.986832	5367.317297	10791.456912	21663.765803	43437.095126
9	153.645237	317.003021	648.339760	1316.468042	2661.939499	5365.619452	10788.148510	21658.083765	43432.096149	87021.036972
10	316.483119	647.337442	1315.700014	2660.315174	5362.722811	10785.711309	21653.405481	43424.120673	87013.958608	174251.803866

Table 5: Lower bounds LB for $n = 12$, $\omega = 0.25$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.043841	0.081972	0.130989	0.190865	0.266944	0.363037	0.480529	0.628061	0.812732	1.039057
2	0.070632	0.117918	0.178455	0.252539	0.346425	0.464788	0.609777	0.791634	1.019109	1.298372
3	0.105556	0.164204	0.239016	0.330725	0.446680	0.592629	0.771711	0.996116	1.276646	1.621585
4	0.150728	0.223479	0.315990	0.429570	0.572890	0.753030	0.974408	1.251588	1.597923	2.024388
5	0.208792	0.299053	0.413515	0.554244	0.731516	0.954059	1.227937	1.570617	1.998619	2.526343
6	0.283044	0.395052	0.536753	0.711198	0.930611	1.205772	1.544850	1.968866	2.498282	3.151839
7	0.377605	0.516628	0.692142	0.908472	1.180219	1.520709	1.940795	2.465863	3.121285	3.931288
8	0.497613	0.670206	0.887713	1.156096	1.492862	1.914503	2.435286	3.085964	3.898020	4.902617
9	0.649485	0.863805	1.133484	1.466580	1.884159	2.406654	3.052657	3.859540	4.866398	6.113139
10	0.841225	1.107427	1.441951	1.855526	2.373589	3.021479	3.823263	4.824479	6.073710	7.621877

Table 6: Lower bounds LB for $n = 12$, $\omega = 0.50$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.072678	0.161743	0.303765	0.512260	0.835131	1.329017	2.057545	3.160770	4.823886	7.289838
2	0.136136	0.270290	0.481933	0.794455	1.275747	2.009575	3.096255	4.739231	7.214110	10.892907
3	0.240030	0.442358	0.758653	1.227697	1.946619	3.039652	4.663051	7.114099	10.803597	16.298576
4	0.406603	0.711871	1.185437	1.889865	2.965256	4.596270	7.024155	10.685454	16.193261	24.408793
5	0.669627	1.130140	1.839987	2.898225	4.508362	6.945374	10.579269	16.053712	24.284623	36.576475
6	1.080233	1.774630	2.839364	4.429203	6.841509	10.486345	15.928368	24.119806	36.430094	54.830912
7	1.715676	2.762123	4.359748	6.748037	10.363638	15.818776	23.971862	36.235452	54.658374	82.215869
8	2.692488	4.268471	6.666092	10.253277	15.673821	23.842630	36.060854	54.428533	82.012528	123.296667
9	4.186227	6.558236	10.156607	15.543532	23.671410	35.908483	54.222503	81.741151	123.057063	184.920777
10	6.461111	10.029172	15.429507	23.517612	35.706258	54.042875	81.498060	122.736677	184.638486	277.358609

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

Table 7: Lower bounds LB for $n = 12$, $\omega = 0.75$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.084765	0.232213	0.521449	1.029109	1.969663	3.677356	6.667402	12.002552	21.459878	38.008935
2	0.192055	0.461949	0.978697	1.890452	3.559701	6.568485	11.846445	21.227391	37.815081	66.862054
3	0.410697	0.902719	1.826188	3.458629	6.418270	11.720383	21.028240	37.518303	66.615075	117.582960
4	0.837312	1.729172	3.376712	6.289308	11.528605	20.867594	37.264250	66.236241	117.268318	206.667197
5	1.645705	3.252837	6.184896	11.364064	20.622759	37.059545	65.912166	116.784759	206.266382	363.015655
6	3.146327	6.026731	11.230987	20.412834	36.746988	65.651335	116.371383	205.649171	362.505104	637.249661
7	5.890822	11.029048	20.243233	36.479173	65.252338	116.039062	205.121913	361.717334	636.599377	1118.015826
8	10.855633	19.985414	36.263037	64.910685	115.529738	204.698535	361.044852	635.593955	1117.187628	1960.524060
9	19.764152	35.933887	64.635264	115.093909	204.048403	360.505506	634.736294	1115.904471	1959.469351	3436.494940
10	35.651590	64.215063	114.742965	203.492467	359.675670	634.049265	1114.810697	1957.831806	3435.151874	6021.575242

Table 8: Lower bounds LB for $n = 12$, $\omega = 1.00$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.077729	0.284624	0.786500	1.835338	4.123417	8.961093	18.808023	39.049886	80.283212	163.315827
2	0.233306	0.698761	1.769008	3.995801	8.743618	18.642097	38.732682	79.744345	162.901078	330.129757
3	0.627968	1.648056	3.904182	8.567588	18.342320	38.503531	79.306844	162.158331	329.557058	665.796128
4	1.550398	3.737447	8.441041	18.099514	38.090311	78.990386	161.554922	328.533308	665.005352	1340.176621
5	3.602732	8.211197	17.924729	37.755405	78.420803	161.117908	327.701095	663.594309	1339.084761	2693.455604
6	8.025368	17.607894	37.514004	77.958870	160.332806	327.097617	662.446552	1337.139934	2691.948071	5406.663567
7	17.351563	37.077261	77.625475	159.695679	326.015462	661.613230	1335.557023	2689.267571	5404.582179	10842.808688
8	36.723686	77.023453	159.235248	325.136715	660.121650	1334.406355	2687.084566	5400.887773	10839.935090	21729.259600
9	76.535753	158.405411	324.500865	658.909675	1332.350475	2685.495752	5397.877229	10834.843333	21725.292384	43522.679728
10	157.732722	323.357020	658.031603	1330.678944	2682.662123	5395.683502	10830.691624	21718.274846	43517.202853	87139.133723

Table 9: Lower bounds LB for $n = 24$, $\omega = 0.25$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.046067	0.086288	0.136525	0.196211	0.269967	0.360632	0.468310	0.599961	0.760489	0.951563
2	0.070741	0.118800	0.178654	0.249867	0.337719	0.445600	0.573966	0.730860	0.922177	1.150399
3	0.101776	0.159246	0.230639	0.315709	0.420509	0.549093	0.702391	0.889729	1.118213	1.391371
4	0.140605	0.209388	0.294651	0.396405	0.521617	0.675150	0.858551	1.082677	1.356116	1.683738
5	0.188975	0.271381	0.373344	0.495220	0.645061	0.828715	1.048530	1.317199	1.645122	2.038884
6	0.249028	0.347863	0.469965	0.616153	0.795765	1.015852	1.279799	1.602499	1.996585	2.470817
7	0.323384	0.442061	0.588493	0.764105	0.979767	1.244006	1.561538	1.949904	2.424482	2.996795
8	0.415255	0.557937	0.733811	0.945094	1.204488	1.522333	1.905039	2.373354	2.946038	3.638105
9	0.528581	0.700350	0.911914	1.166514	1.479053	1.862096	2.324214	2.890039	3.582504	4.421032
10	0.668201	0.875270	1.130173	1.437462	1.814696	2.277175	2.836214	3.521169	4.360122	5.378058

Table 10: Lower bounds LB for $n = 24$, $\omega = 0.50$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.074179	0.161738	0.293387	0.474986	0.741980	1.131071	1.674408	2.462833	3.605201	5.226780
2	0.126445	0.248344	0.430845	0.684806	1.058100	1.602911	2.370222	3.486995	5.110986	7.433504
3	0.206441	0.377366	0.632400	0.990220	1.516275	2.285340	3.377045	4.970647	7.296035	10.643990
4	0.327616	0.568905	0.928001	1.435685	2.182479	3.276271	4.840111	7.129419	10.480790	15.334657
5	0.509838	0.852614	1.361816	2.086798	3.154149	4.720472	6.974443	10.282977	15.140910	22.213605
6	0.782486	1.272311	1.999099	3.040553	4.575482	6.832408	10.098987	14.906060	21.983595	32.334813
7	1.189050	1.892832	2.936435	4.440616	6.660268	9.930364	14.687625	21.704775	32.061755	47.269020
8	1.793981	2.810267	4.317006	6.500151	9.725992	14.487439	21.445449	31.730735	46.944859	69.359540
9	2.692905	4.167211	6.353400	9.535897	14.244800	21.207792	31.422863	46.551868	68.974714	102.105356
10	4.027874	6.175555	9.361674	14.019116	20.919724	31.140725	46.186366	68.508153	101.648515	150.734982

Table 11: Lower bounds LB for $n = 24$, $\omega = 0.75$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.081994	0.213943	0.451756	0.832248	1.494796	2.634776	4.514153	7.738492	13.265709	22.605900
2	0.157833	0.372165	0.756541	1.383750	2.477074	4.364877	7.519089	12.953703	22.312198	38.256878
3	0.300480	0.654833	1.287099	2.335251	4.163411	7.328598	12.673595	21.913746	37.882271	65.262335
4	0.563256	1.157136	2.211873	3.982294	7.071243	12.430534	21.556167	37.373459	64.784590	112.014586
5	1.040154	2.045815	3.824811	6.839962	12.101808	21.246059	36.917023	64.134902	111.405377	193.158061
6	1.896391	3.612648	6.638966	11.806493	20.826198	36.521415	63.552330	110.575876	192.381302	334.263625
7	3.421799	6.367912	11.549983	20.449150	35.985192	63.047703	109.832379	191.322310	333.273358	580.003449
8	6.124171	11.203717	20.121827	35.503830	62.362921	109.188759	190.373524	331.921498	578.741144	1008.449988
9	10.892449	19.679508	35.086185	61.748438	108.314327	189.552721	330.710857	577.015574	1006.841122	1756.089696
10	19.282035	34.521208	61.215600	107.529977	188.436203	329.664214	575.470965	1004.638728	1754.039395	3061.584216

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106

Table 12: Lower bounds LB for $n = 24$, $\omega = 1.00$, and various m_1, m_2

$m_1 \backslash m_2$	1	2	3	4	5	6	7	8	9	10
1	0.066583	0.228412	0.585928	1.257210	2.648840	5.473483	10.956983	21.961490	43.955480	87.373766
2	0.154876	0.469429	1.153739	2.473382	5.194556	10.712378	21.543912	43.289086	86.797750	173.081323
3	0.368817	0.994240	2.332056	4.954606	10.330652	21.210059	42.718331	85.886164	172.295786	344.167808
4	0.856593	2.113702	4.761602	10.002539	20.687691	42.262736	85.106135	171.048913	343.096731	685.966058
5	1.925404	4.462699	9.738997	20.239068	41.547970	84.484501	169.983000	341.391414	684.505914	1369.141686
6	4.205138	9.329863	19.879259	40.934643	83.506561	169.134954	339.935018	682.173827	1367.151522	2735.028246
7	8.977595	19.319288	40.443471	82.668151	167.797060	338.778288	680.184145	1363.962629	2732.316198	5466.297941
8	18.837535	39.677117	81.997758	166.651093	336.948119	678.606645	1361.244735	2727.956163	5462.602903	10928.302889
9	39.018352	80.949044	165.736225	335.381953	676.103307	1359.093786	2724.244020	5456.642254	10923.269616	21851.756323
10	80.048320	164.301240	334.133652	673.963116	1355.670007	2721.311699	5451.572831	10915.121643	21844.901609	43698.091384

6 Conclusion and Future Work

In this paper we have established sharp lower bounds for the projection weighted symmetric discrepancy (PWSDisc) of mixed two- and three-level balanced designs. Our work extends the recent results of [25] to the practically important class of mixed-level designs, which are ubiquitous in industrial and scientific experiments. The main contributions are summarized as follows: We derived an explicit formula for PWSDisc in terms of coincidence counts and row profiles. This expression isolates the contributions of two-level and three-level factors and reveals the combinatorial structure underlying the discrepancy. Using a combinatorial optimization tool and Jensen's inequality, we obtained a lower bound for PWSDisc that is easy to compute and demonstrably tight. The bound respects the integrality of the pattern counts and is applicable for any weight. Special cases for symmetric discrepancy, pure three-level designs, and pure two-level designs were given. We demonstrated how the lower bound can be integrated into a threshold accepting algorithm as a principled termination criterion. This reduces computational cost and provides a quality guarantee for the resulting designs. A comprehensive catalog of lower bounds for various parameter combinations offers immediate benchmarks for practitioners.

The lower bounds derived in this paper serve two essential roles in uniform design theory and practice. First, they provide a benchmark for assessing design uniformity: a design whose PWSDisc value equals or approaches the lower bound is provably optimal or near-optimal. Second, they act as termination criteria for stochastic search algorithms, enabling efficient construction of uniform designs without wasteful over-iteration. The threshold accepting algorithm illustrates this concept and can be readily adapted to other optimization heuristics. Moreover, the catalog of lower bounds for various parameter combinations is a useful reference for experimenters who need to quickly evaluate the quality of a candidate design or to set realistic targets when planning a design search.

While our results cover a wide range of mixed two- and three-level designs, several directions remain open for further investigation.

1. **Extension to higher numbers of levels:** The pattern-counting methodology developed here can, in principle, be applied to designs mixing three or more distinct level counts (e.g., two, three, and four levels simultaneously). However, the number of distinct kernel values grows rapidly, making manual enumeration cumbersome. Automating the classification and the derivation of the linear constraints would be a valuable contribution. One might also seek closed-form expressions depending only on a few summary statistics, such as the moments of the level distributions.

2. **Other mixed-level combinations:** Beyond two and three levels, many practical experiments involve mixtures such as two and four levels, or three and four levels. Extending our results to those cases would broaden the applicability of the theory and provide benchmarks for a wider class of designs.

3. **Average discrepancy under level permutations:** Recent studies [33, 34] have considered the average discrepancy over all level permutations of a design. Investigating the average PWSDisc

and its lower bounds would provide a different perspective on design uniformity and might yield designs that are robust to level coding. Moreover, the average PWSDisc can be used to detect isomorphic designs [35, 36], which is a critical problem in design of experiment theory.

4. **Algorithmic enhancements:** The threshold accepting algorithm can be further refined by incorporating the necessary optimality conditions (e.g., biasing neighbour generation toward configurations that satisfy the extremal distribution of the τ_i counts). Hybrid approaches combining threshold accepting with local search or genetic algorithms merit exploration. Additionally, parallel implementations could accelerate the search for very large designs.

5. **Foldover designs:** The foldover technique is widely used for breaking confounding among factorial effects [37, 38]. Studying the construction of foldover (follow-up) designs using PWSDisc is an interesting topic for future research.

Acknowledgments

The authors sincerely thank the referees, Associate Editor, and Editor-in-Chief for their valuable comments and suggestions, which have greatly improved this paper. The authors also acknowledge the use of DeepSeek for assistance in improving the English grammar and language clarity.

Funding

This work was supported by the National Nature Science Foundation of China No. (12371261).

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] S.M. Celem, B. Barkahoum, G.K. Vishwakarma, and H. Qin, "Advances in uniform experimental designs: A decade selective review of algorithmic search and deterministic construction methods," *Mathematical Applications and Statistical Rigor*, vol. 1(1), p. 1–56, 2026.

[2] A.M. Elsawah, "A novel hybrid algorithm for designing mixed three-and nine-level experiments without modeling assumptions," *Communications in Statistics—Simulation and Computation*, vol. 54(4), p. 1003–1037, 2023.

[3] B.B. Prasath, A.M. Elsawah, Z. Liyuan, and K. Poon, "Modeling and optimization of the effect of abiotic stressors on the productivity of the biomass, chlorophyll and lutein in microalgae Chlorella pyrenoidosa," *Journal of Agriculture and Food Research*, vol. 5, p. 100163, 2021.

[4] K.T. Fang, "Uniform design: application of number-theoretic methods in experimental design," *Acta Mathematicae Applicatae Sinica*, vol. 3, p. 363–372, 1980.

[5] Y. Wang and K.T. Fang, "A note on uniform distribution and experimental design," *Chinese Science Bulletin*, vol. 26, p. 485–489, 1981.

[6] K.T. Fang, R.Z. Li, and A. Sudjianto, *Design and Modeling for Computer Experiments*. New York: Chapman and Hall/CRC, 2006.

[7] A.M. Elsawah, "Constructing orthogonal maximin distance uniform projection designs for computer experiments," *Journal of Computational and Applied Mathematics*, vol. 473, p. 116902, 2026.

[8] F.J. Hickernell, "Goodness-of-fit statistics, discrepancies and robust designs," *Statistics & Probability Letters*, vol. 44, p. 73–78, 1999.

[9] R.X. Yue and F.J. Hickernell, "Robust designs for fitting linear models with misspecification," *Statistica Sinica*, vol. 9, p. 1053–1069, 1999.

[10] A.M. Elsawah, "Designing uniform computer sequential experiments with mixture levels using lee discrepancy," *Journal of Systems Science and Complexity*, vol. 32, p. 681–708, 2019.

[11] F.J. Hickernell, "A generalized discrepancy and quadrature error bound," *Mathematics of Computation*, vol. 67, p. 299–322, 1998.

[12] H. Qin and K.T. Fang, "Discrete discrepancy in factorial designs," *Metrika*, vol. 60, p. 59–72, 2004.

[13] Y.D. Zhou, J.H. Ning, and X.B. Song, "Lee discrepancy and its applications in experimental designs," *Statistics & Probability Letters*, vol. 78, p. 1933–1942, 2008.

[14] Y.D. Zhou, K.F. Fang, and J.H. Ning, "Mixture discrepancy for quasi-random point sets," *Journal of Complexity*, vol. 29, p. 283–301, 2013.

[15] L. He, M. Xie, and J. Ning, "Projection weighted symmetric discrepancy (in Chinese)," *Scientia Sinica Mathematica*, vol. 50, p. 629–644, 2020.

[16] K.T. Fang, D. Maringer, Y. Tang, and P. Winker, "Lower bounds and stochastic optimization algorithms for uniform designs with three or four levels," *Mathematics of Computation*, vol. 75, p. 859–878, 2005.

[17] K.T. Fang, X. Lu, and P. Winker, "Lower bounds for centered and wrap-around L_2-discrepancies and construction of uniform designs by threshold accepting," *Journal of Complexity*, vol. 19, p. 692–711, 2003.

[18] X. Ke, R. Zhang, and H.J. Ye, "Two- and three-level lower bounds for mixture L_2-discrepancy and construction of uniform designs by threshold accepting," *Journal of Complexity*, vol. 31(5), p. 741–753, 2015.

[19] A.M. Elsawah, K.T. Fang, P. He, and H. Qin, "Sharp lower bounds of various uniformity criteria for constructing uniform designs," *Statistical Papers*, vol. 62(1), p. 1461–1482, 2021.

[20] Y.J. Lei, H. Qin, and N. Zou, "Some lower bounds of centered L_2-discrepancy on foldover designs," *Acta Mathematica Scientia*, vol. 30A(6), p. 1555–1561, 2010.

[21] A.M. Elsawah and H. Qin, "An efficient methodology for constructing optimal foldover designs in terms of mixture discrepancy," *Journal of the Korean Statistical Society*, vol. 45, p. 77–88, 2016.

[22] Y.D. Zhou, H. Xu, and B. Tang, "A lower bound for centered L_2 discrepancy on asymmetric factorials and its applications," *Science China Mathematics*, vol. 56, p. 1027–1038, 2013.

[23] Z. Wang, H. Qin, and K. Chatterjee, "Lower bounds on the symmetric L_2-discrepancy and their application," *Communications in Statistics—Theory and Methods*, vol. 36, p. 2413–2423, 2007.

[24] Y. Lei and Z. Ou, "Lower bounds for the symmetric L_2-discrepancy of U-type designs (in Chinese)," *Acta Mathematica Scientica*, vol. 42A, p. 1802–1811, 2022.

[25] H. Zheng, K. Fu, and Y. Xiao, "Lower bounds of projection weighted symmetric discrepancy on uniform designs," *Australian & New Zealand Journal of Statistics*, vol. 67(1), p. 104–120, 2025.

[26] A.M. Elsawah and H. Qin, "Lower bound of centered L_2-discrepancy for mixed two and three levels U-type designs," *Journal of Statistical Planning and Inference*, vol. 161, p. 1–11, 2015.

[27] S.M. Celem and H. Qin, "Lower bounds of unanchored discrepancy for mixed-level U-type designs," *Mathematical Applications and Statistical Rigor*, vol. 1(1), p. 70–88, 2026.

[28] P. Winker and K.T. Fang, "Optimal U-type design," in *Monte Carlo and Quasi-Monte Carlo Methods 1996*, p. 436–448. Springer, 1998.

[29] P. Winker and K.T. Fang, "Optimal U-type design," in *Monte Carlo and Quasi-Monte Carlo Methods 1996*, p. 436–448. Springer, 1998.

[30] K.T. Fang, X. Ke, and A.M. Elsawah, "Construction of uniform designs via an adjusted threshold accepting algorithm," *Journal of Complexity*, vol. 43, p. 28–37, 2017.

[31] C.J. Wu and M.S. Hamada, *Experiments: Planning, Analysis, and Optimization*, Third Edition. John Wiley & Sons, 2021.

[32] A.M. Elsawah and H. Qin, "A new strategy for optimal foldover two-level designs," *Statistics & Probability Letters*, vol. 103, p. 116–126, 2015.

[33] A.M. Elsawah, "Improving the space-filling behavior of multiple triple designs," *Computational and Applied Mathematics*, vol. 41, p. 180, 2022.

[34] A.M. Elsawah, "Level permutations and factor projections of multiple quadruple designs," *Communications in Statistics—Simulation and Computation*, vol. 53(10), p. 4893–4920, 2024.

[35] X. Ke, K.T. Fang, A.M. Elsawah, and Y. Lin, "New non-isomorphic detection methods for orthogonal designs," *Communications in Statistics—Simulation and Computation*, vol. 52(1), p. 27–42, 2023.

[36] L.C. Weng, K.T. Fang, and A.M. Elsawah, "Degree of isomorphism: a novel criterion for identifying and classifying orthogonal designs," *Statistical Papers*, vol. 64, p. 93–116, 2023.

[37] A.M. Elsawah, "Novel techniques for performing successful follow-up experiments based on prior information from initial-stage experiments," *Statistics*, vol. 56(5), p. 1133–1165, 2022.

[38] A.M. Elsawah, "Choice of optimal second stage designs in two-stage experiments," *Computational Statistics*, vol. 33, p. 933–965, 2018.

DOI: 10.1234/sirg.esit.x.x.x Celem et al., 2026, MASR, VOL. 1,NO. 2, PP. 89–106