

Mathematical Applications and Statistical Rigor MASR

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Journal Homepage

<https://pub.scientificirg.com/index.php/MASR/index>



Likelihood and Bayesian Inference for Exponentiated Lomax Stress-Strength Reliability Under Progressive Type-II Censoring with Binomial Removals

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ABSTRACT

This paper develops frequentist and Bayesian inference for the stress-strength (SS) reliability $\delta = P(W < V)$, where the strength V and the stress W are independent Exponentiated Lomax (EL) random variables with common shape and scale parameters (β, θ) but distinct exponentiation parameters α_1 and α_2 . The lifetime data are progressively Type-II censored (PT-IIC) with binomial removals. It is shown that $\delta = \alpha_1 / (\alpha_1 + \alpha_2)$. Maximum likelihood estimators (MLEs) of the model parameters and of δ are derived from the analytic score equations, closed-form estimators of the removal probabilities are obtained, and asymptotic confidence intervals are constructed by the delta method. Bayes estimators of δ are developed under the squared error and linear-exponential (LINEX) loss functions, using informative and non-informative gamma priors and a Metropolis-Hastings sampler. In a Monte Carlo study of 128 scenarios with 300 replications each, the root mean squared errors (RMSEs) of all estimators decrease as the sample size n and the number of observed failures m increase. Bayes estimators with informative priors elicited from pilot samples attain the smallest RMSE in all 128 scenarios (average 0.040 versus 0.052 for the MLE), whereas Bayes estimators with non-informative priors perform similarly to the MLE (average RMSE 0.051). Because the informative priors are built from samples generated under the true parameter values, the resulting advantage is an optimistic upper bound on the benefit of reliable prior information. The methods are illustrated with two classical data sets of insulating-fluid breakdown times at 34 kV and 36 kV. From PT-IIC samples with $m = 10$ per data set, the MLE of δ is 0.644 (95% asymptotic interval 0.462 to 0.792) and the Bayes estimate under squared error loss is 0.632.

PAPER INFORMATION

HISTORY

Received: 30 July 2026

Revised: 28 August 2026

Accepted: 08 October 2026

Online: 25 October 2026

MSC

62N05; 62F15; 62F10; 62N02; 62F03

KEYWORDS

Likelihood;
Bayesian Inference;
Exponentiated Lomax;
Binomial Removals;
Metropolis-Hastings
Algorithm.

1. INTRODUCTION

Reliability analysis supports decision making in engineering, medicine, and defense applications. Among the models used in this field, the stress-strength (SS) model, defined by $\delta = P(W < V)$, has attracted sustained attention. Here V denotes the strength of a component or system, W denotes the stress acting on it, and δ is the probability that the component operates without failure. Applications range from the failure analysis of mechanical structures and aircraft windshields to clinical trials and actuarial risk assessment [1].

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Classical and Bayesian estimation of δ has been studied for the Burr XII, generalized exponential, generalized Pareto, Weibull, exponential, inverse Weibull, bathtub-shaped, exponentiated exponential, unit-half-normal, and unit Gompertz distributions, among others, under complete, censored, and ranked set sampling [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. More recent contributions address the exponentiated generalized Marshall-Olkin-G, Marshall-Olkin extended Weibull, inverse Pareto, Chen, and Burr III models [13, 14, 15, 16, 17].

A common censoring scheme in life testing is progressive Type-II censoring (PT-IIC), in which surviving units are withdrawn at random from the experiment at each observed failure time according to a removal scheme $R = (R_1, R_2, \dots, R_m)$ [18, 19]. When the removals are random and binomially distributed, the scheme is called PT-IIC with binomial removals. This setting is more realistic in clinical and industrial studies, where withdrawals cannot be fully controlled [20, 21].

The Exponentiated Lomax (EL) distribution generalizes the Lomax (Pareto type-II) distribution and is attractive in lifetime and reliability modeling because of its tractable form and flexible heavy-tailed shape [22, 23, 24, 25, 26]. Stress-strength reliability under PT-IIC with binomial removals has been examined for the Burr XII and Burr III distributions with a common shape parameter [27], but, to the best of the author's knowledge, the EL case has not been considered. The present paper fills this gap by exploiting the common shape and scale parameters to obtain a closed-form reliability expression, while retaining the full four-parameter likelihood for estimation and inference. Because the closed form follows directly from the shared baseline distribution, the emphasis is placed on the inferential methods under binomial removals and on their practical performance.

The main contributions of this paper are as follows.

1. A closed-form expression for $\delta = P(W < V)$ is derived when V and W are independent EL variables sharing the parameters β and θ .
2. Maximum likelihood estimators (MLEs) of all model parameters and of δ are obtained from PT-IIC data with binomial removals, together with asymptotic confidence intervals.
3. Bayes estimators of δ are obtained under the squared error loss function (SELF) and the LINEX loss function with informative (INF) and non-informative (N-INF) gamma priors by Metropolis-Hastings (MH) Markov chain Monte Carlo (MCMC) sampling.
4. A Monte Carlo study compares all estimators across sample sizes, censoring fractions, removal probabilities, and removal schemes.
5. The methodology is applied to two real data sets of insulating-fluid breakdown times.

The remainder of the paper is organized as follows. Section 2 describes the EL distribution and derives the SS reliability formula. Section 3 develops maximum likelihood estimation under PT-IIC with binomial removals. Section 4 presents the Bayesian framework and the MH algorithm. Section 5 reports the Monte Carlo study. Section 6 analyzes the real data, and Section 7 concludes.

2. THE EXPONENTIATED LOMAX DISTRIBUTION AND THE SS RELIABILITY FORMULA

2.1 The Exponentiated Lomax Distribution

Let $G(x) = 1 - (1 + x/\theta)^{-\beta}$, $x > 0$, denote the Lomax cumulative distribution function (CDF) with shape $\beta > 0$ and scale $\theta > 0$. A random variable X follows the EL distribution with shape parameters $\alpha > 0$, $\beta > 0$ and scale parameter $\theta > 0$, written $X \sim \text{EL}(\alpha, \beta, \theta)$, if its CDF is $F(x) = [G(x)]^\alpha$. The probability density function (PDF) is

$$f(x; \alpha, \beta, \theta) = \frac{\alpha\beta}{\theta} \left(1 + \frac{x}{\theta}\right)^{-\beta-1} \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}\right]^{\alpha-1}, \quad x > 0, \quad (1)$$

and the CDF, survival function (SF), and hazard rate function (HRF) are, respectively,

$$F(x) = \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}\right]^\alpha, \quad (2)$$

$$\bar{S}(x) = 1 - \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}\right]^\alpha, \quad (3)$$

$$h(x) = \frac{f(x)}{\bar{S}(x)} = \frac{\frac{\alpha\beta}{\theta} \left(1 + \frac{x}{\theta}\right)^{-\beta-1} \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}\right]^{\alpha-1}}{1 - \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}\right]^{\alpha}}. \quad (4)$$

The PDF in **Equation 1** and the CDF in **Equation 2** give the SF in **Equation 3** and the HRF in **Equation 4**. The case $\alpha = 1$ reduces to the Lomax distribution. When $\theta \rightarrow \infty$ and $\beta \rightarrow \infty$ with $\beta/\theta \rightarrow \lambda$, the CDF tends to $(1 - e^{-\lambda x})^{\alpha}$, that of the exponentiated exponential distribution. **Figures 1 and 2** display the PDF and HRF for selected parameter values with $\theta = 1$. For $\alpha < 1$ the density is unbounded at the origin, whereas for $\alpha > 1$ it starts at zero and is unimodal in the cases displayed in **Figure 1**. Numerical inspection of the HRF over a wide grid of parameter values indicates that it is decreasing for $\alpha \leq 1$ and upside-down bathtub shaped (increasing, then decreasing) for $\alpha > 1$; in both cases $h(x) \rightarrow 0$ as $x \rightarrow \infty$, as shown in **Figure 2**. Since $F(x)$ is decreasing in α for fixed x , a larger α makes the variable stochastically larger, which links α to the strength of a component.

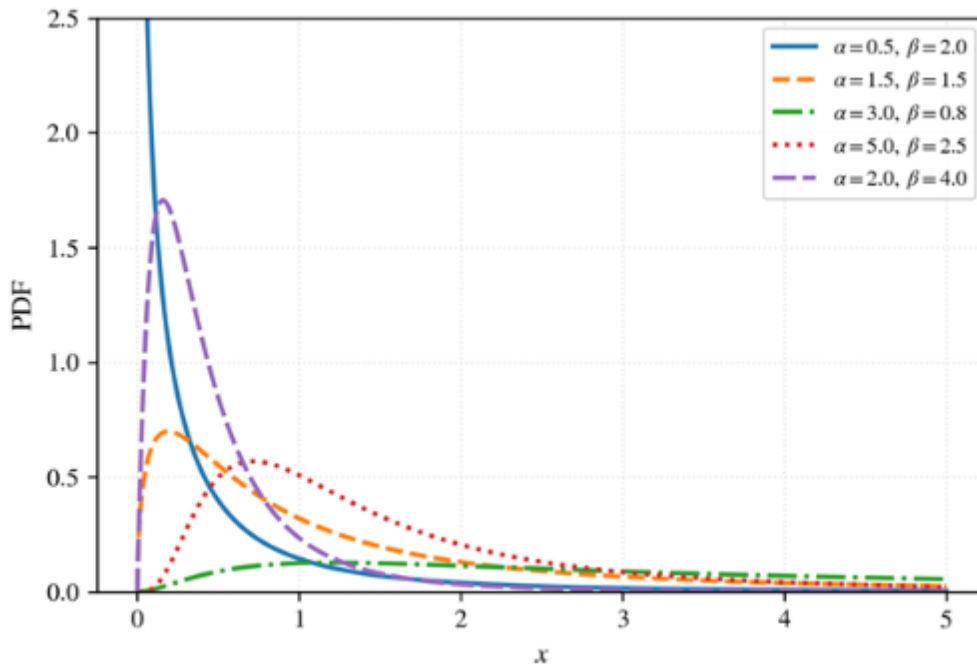


Figure 1: PDF of the EL distribution for selected values of (α, β) with $\theta = 1$

2.2 The SS Reliability Formula

Let the strength $V \sim \text{EL}(\alpha_1, \beta, \theta)$ and the stress $W \sim \text{EL}(\alpha_2, \beta, \theta)$ be independent, sharing β and θ but having distinct exponentiation parameters α_1 and α_2 . The SS reliability is

$$\delta = P(W < V) = \int_0^\infty f_V(v) F_W(v) dv. \quad (5)$$

Since $F_V = G^{\alpha_1}$ and $F_W = G^{\alpha_2}$, the substitution $u = G(v)$ in **Equation 5** gives $f_V(v) dv = \alpha_1 u^{\alpha_1-1} du$ and $F_W(v) = u^{\alpha_2}$, hence

$$\delta = \alpha_1 \int_0^1 u^{\alpha_1+\alpha_2-1} du = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \quad (6)$$

Thus δ depends only on α_1 and α_2 and is free of the common parameters β and θ . In addition, $\delta > 1/2$ if and only if $\alpha_1 > \alpha_2$, and δ increases with α_1 and decreases with α_2 . The closed form simplifies inference, but it requires β and θ to be common to the two populations.

The closed-form expression in **Equation 6** relies on the assumption that the stress and strength populations share the Lomax shape β and scale θ . This restriction is substantive: if β or θ differs between the populations, δ must generally be obtained by numerical integration of **Equation 5**. In practice, the assumption should therefore be assessed before the proposed estimators are applied, for example through a likelihood ratio test or a model comparison procedure.

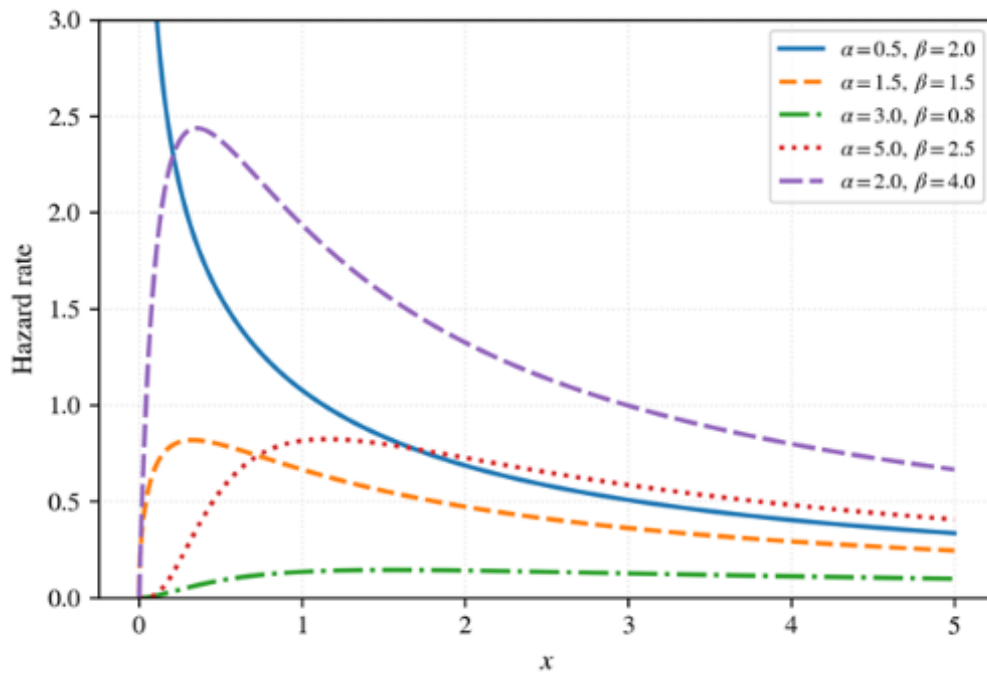


Figure 2: HRF of the EL distribution for selected values of (α, β) with $\theta = 1$

3. MAXIMUM LIKELIHOOD ESTIMATION UNDER PT-IIC WITH BINOMIAL REMOVALS

Let $v_1 < \dots < v_{m_1}$ be a PT-IIC sample of size m_1 from $EL(\alpha_1, \beta, \theta)$ obtained from n_1 units under the scheme $R = (R_1, \dots, R_{m_1})$, and let $w_1 < \dots < w_{m_2}$ be an independent PT-IIC sample of size m_2 from $EL(\alpha_2, \beta, \theta)$ obtained from n_2 units under the scheme $R^\circ = (R_1^\circ, \dots, R_{m_2}^\circ)$, with $\sum_i R_i = n_1 - m_1$ and $\sum_j R_j^\circ = n_2 - m_2$. Conditionally on the removal vectors, the likelihood function of $(\alpha_1, \alpha_2, \beta, \theta)$ is

$$L(\alpha_1, \alpha_2, \beta, \theta \mid \text{data}) = K_1 K_2 \prod_{i=1}^{m_1} f_V(v_i) [\bar{S}_V(v_i)]^{R_i} \prod_{j=1}^{m_2} f_W(w_j) [\bar{S}_W(w_j)]^{R_j^\circ}, \quad (7)$$

where K_1 and K_2 are constants that depend only on the sample sizes and the removal schemes and do not involve the parameters. Under binomial removal, the distribution of the removal vector involves only the removal probabilities, so the full likelihood factorizes into the lifetime part above and two binomial parts for p_1 and p_2 . The lifetime parameters and δ can therefore be estimated without reference to p_1 and p_2 [28, 29, 30, 31].

With $G_i = G(v_i)$ and $G_j^\circ = G(w_j)$, the log-likelihood $\ell = \ln L$ is, up to an additive constant, $\ell = \ell_V + \ell_W$, where the two components, derived from **Equation 7** and **Equation 1**, are

$$\ell_V = m_1 \ln \alpha_1 + m_1 \ln \frac{\beta}{\theta} - (\beta + 1) \sum_{i=1}^{m_1} \ln \left(1 + \frac{v_i}{\theta} \right) + (\alpha_1 - 1) \sum_{i=1}^{m_1} \ln G_i + \sum_{i=1}^{m_1} R_i \ln (1 - G_i^{\alpha_1}), \quad (8)$$

$$\ell_W = m_2 \ln \alpha_2 + m_2 \ln \frac{\beta}{\theta} - (\beta + 1) \sum_{j=1}^{m_2} \ln \left(1 + \frac{w_j}{\theta} \right) + (\alpha_2 - 1) \sum_{j=1}^{m_2} \ln G_j^\circ + \sum_{j=1}^{m_2} R_j^\circ \ln (1 - (G_j^\circ)^{\alpha_2}). \quad (9)$$

Let $s_i = \ln(1 + v_i/\theta)$, $A_i = (1 + v_i/\theta)^{-\beta} = 1 - G_i$, and $k_i = v_i/[\theta(\theta + v_i)]$, and define s_j° , A_j° , and k_j° analogously from w_j . The score equations are obtained by differentiating **Equation 8** and **Equation 9**:

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{m_1}{\alpha_1} + \sum_{i=1}^{m_1} \ln G_i - \sum_{i=1}^{m_1} \frac{R_i G_i^{\alpha_1} \ln G_i}{1 - G_i^{\alpha_1}} = 0, \quad (10)$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{m_2}{\alpha_2} + \sum_{j=1}^{m_2} \ln G_j^\circ - \sum_{j=1}^{m_2} \frac{R_j^\circ (G_j^\circ)^{\alpha_2} \ln G_j^\circ}{1 - (G_j^\circ)^{\alpha_2}} = 0, \quad (11)$$

with score weights

$$\eta_i(\alpha_1) = (\alpha_1 - 1) - \frac{R_i \alpha_1 G_i^{\alpha_1}}{1 - G_i^{\alpha_1}}, \quad \eta_j^\circ(\alpha_2) = (\alpha_2 - 1) - \frac{R_j^\circ \alpha_2 (G_j^\circ)^{\alpha_2}}{1 - (G_j^\circ)^{\alpha_2}}, \quad (12)$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{m_1 + m_2}{\beta} - \sum_{i=1}^{m_1} s_i - \sum_{j=1}^{m_2} s_j^\circ + \sum_{i=1}^{m_1} \eta_i \frac{A_i s_i}{G_i} + \sum_{j=1}^{m_2} \eta_j^\circ \frac{A_j^\circ s_j^\circ}{G_j^\circ} = 0, \quad (13)$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{m_1 + m_2}{\theta} + (\beta + 1) \left[\sum_{i=1}^{m_1} k_i + \sum_{j=1}^{m_2} k_j^\circ \right] - \beta \left[\sum_{i=1}^{m_1} \eta_i \frac{A_i k_i}{G_i} + \sum_{j=1}^{m_2} \eta_j^\circ \frac{A_j^\circ k_j^\circ}{G_j^\circ} \right] = 0. \quad (14)$$

When there are no removals ($R_i = R_j^\circ = 0$), the weights in **Equation 12** reduce to $\eta_i = \alpha_1 - 1$, and for given (β, θ) the solutions of **Equation 10** and **Equation 11** are available in closed form, as given in **Equation 15**,

$$\hat{\alpha}_1 = -\frac{m_1}{\sum_{i=1}^{m_1} \ln G_i}, \quad \hat{\alpha}_2 = -\frac{m_2}{\sum_{j=1}^{m_2} \ln G_j^\circ}. \quad (15)$$

In the general case, the system in **Equation 10** to **Equation 14** has no closed-form solution and must be solved numerically. The log-likelihood in **Equation 8** and **Equation 9** is maximized with the quasi-Newton L-BFGS-B algorithm, using the analytic score vector above, on the logarithmic scale $\boldsymbol{\phi} = (\ln \alpha_1, \ln \alpha_2, \ln \beta, \ln \theta)$ restricted to $[10^{-3}, 10^3]$ for each parameter. Several initial points for (β, θ) with $\alpha_1 = \alpha_2 = 1$ are used, and the solution with the largest log-likelihood is retained. For small samples the likelihood can be nearly flat along the ridge $\beta, \theta \rightarrow \infty$ with β/θ fixed (the exponentiated exponential limit), so $\hat{\beta}$ and $\hat{\theta}$ may reach the upper bound. This does not prevent $\hat{\delta}$ from being estimated stably, because δ depends only on α_1 and α_2 .

Under the binomial removal model, the removal counts satisfy

$$R_1 \sim \text{Bin}(n_1 - m_1, p_1), \quad (16)$$

$$R_i \mid R_1, \dots, R_{i-1} \sim \text{Bin}\left(n_1 - m_1 - \sum_{s=1}^{i-1} R_s, p_1\right), \quad i = 2, \dots, m_1 - 1, \quad (17)$$

with $R_{m_1} = n_1 - m_1 - \sum_{i < m_1} R_i$ determined by the others. The MLE of the removal probability, which is the total number of removals divided by the total number of Bernoulli trials, is

$$\hat{p}_1 = \frac{\sum_{i=1}^{m_1-1} R_i}{(m_1 - 1)(n_1 - m_1) - \sum_{i=1}^{m_1-1} (m_1 - i - 1) R_i}, \quad (18)$$

and $\hat{p}_2$ is obtained analogously from R° . The denominator of **Equation 18** counts the Bernoulli trials and is positive whenever $n_1 > m_1$. By the invariance property of MLEs and **Equation 6**,

$$\hat{\delta} = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 + \hat{\alpha}_2}. \quad (19)$$

Let $\hat{\mathbf{V}} = [-\partial^2 \ell / \partial \boldsymbol{\phi} \partial \boldsymbol{\phi}']^{-1}$, evaluated at $\hat{\boldsymbol{\phi}}$, be the inverse observed information on the logarithmic scale, with elements $\hat{v}_{kl}$. Because logit $\delta = \ln \alpha_1 - \ln \alpha_2 = \phi_1 - \phi_2$, the delta method gives the $100(1 - \gamma)\%$ asymptotic confidence interval (ACI) for δ

$$\left(\frac{1}{1 + e^{-(\hat{\phi}_1 - \hat{\phi}_2 - z_{1-\gamma/2}\hat{\delta})}}, \frac{1}{1 + e^{-(\hat{\phi}_1 - \hat{\phi}_2 + z_{1-\gamma/2}\hat{\delta})}} \right), \quad \hat{s}^2 = \hat{v}_{11} + \hat{v}_{22} - 2\hat{v}_{12}, \quad (20)$$

which always lies in $(0, 1)$. The observed information is obtained by numerical differentiation of the analytic score vector given by **Equation 10**, **Equation 11**, **Equation 13**, and **Equation 14**, transformed to the logarithmic scale.

4. BAYESIAN ESTIMATION

4.1 Prior Distributions

Independent gamma priors, given in **Equation 21**, are adopted for the four positive parameters $\xi_1 = \alpha_1$, $\xi_2 = \alpha_2$, $\xi_3 = \beta$, and $\xi_4 = \theta$:

$$\pi_k(\xi_k) \propto \xi_k^{a_k-1} e^{-b_k \xi_k}, \quad a_k, b_k, \xi_k > 0. \quad (21)$$

In the simulation study, the informative (INF) hyper-parameters (a_k, b_k) are elicited from pilot samples. For each scenario, 100 pseudo-samples are generated, the MLEs are computed, and the prior mean and variance of each parameter are matched

to the median μ_k and the squared scaled interquartile range $\sigma_k^2 = (\text{IQR}/1.349)^2$ of the pilot MLEs, that is, $a_k = \mu_k^2/\sigma_k^2$ and $b_k = \mu_k/\sigma_k^2$. These robust moments are used because the pilot MLEs of β and θ occasionally reach the boundary. For the non-informative (N-INF) prior, all hyper-parameters are set to the small value 10^{-3} , which is a proper approximation to the improper prior $\pi(\xi_k) \propto 1/\xi_k$ obtained when the hyper-parameters are zero. The posterior does not depend on the removal probabilities, which have separate Beta posteriors and are not needed for δ .

4.2 Posterior Distribution and Loss Functions

The joint posterior of $(\alpha_1, \alpha_2, \beta, \theta)$ is proportional to the product of the likelihood in **Equation 7** and the priors in **Equation 21**, as stated in **Equation 22**,

$$\pi(\alpha_1, \alpha_2, \beta, \theta \mid \text{data}) \propto L(\alpha_1, \alpha_2, \beta, \theta \mid \text{data}) \prod_{k=1}^4 \pi_k(\xi_k), \quad (22)$$

which has no closed form, so posterior summaries are computed by MCMC. Under the squared error loss function (SELF), the Bayes estimator of δ is the posterior mean, given in **Equation 24**. The LINEX loss function [32] is

$$L(\Delta) = e^{c\Delta} - c\Delta - 1, \quad \Delta = \tilde{\delta} - \delta, \quad (23)$$

with $c \neq 0$. Positive values of c penalize overestimation more heavily than underestimation, and negative values penalize underestimation more heavily. The Bayes estimators under **Equation 23** are

$$\tilde{\delta}_{\text{SELF}} = E[\delta \mid \text{data}], \quad (24)$$

$$\tilde{\delta}_{\text{LINEX}} = -\frac{1}{c} \ln E[e^{-c\delta} \mid \text{data}]. \quad (25)$$

By Jensen's inequality, $\tilde{\delta}_{\text{LINEX}} \leq \tilde{\delta}_{\text{SELF}}$ for $c > 0$ and $\tilde{\delta}_{\text{LINEX}} \geq \tilde{\delta}_{\text{SELF}}$ for $c < 0$. The values $c = -1.5$ and $c = 1.5$ are used, and the corresponding estimators are denoted $\text{LINEX}_{-1.5}$ and $\text{LINEX}_{1.5}$.

4.3 Metropolis-Hastings Algorithm

A random-walk Metropolis-Hastings algorithm [33] is run on the logarithmic scale ϕ , whose target density, including the Jacobian of the transformation, is

$$\pi(\phi \mid \text{data}) \propto L(e^{\phi_1}, e^{\phi_2}, e^{\phi_3}, e^{\phi_4} \mid \text{data}) \prod_{k=1}^4 \exp(a_k \phi_k - b_k e^{\phi_k}). \quad (26)$$

The algorithm proceeds as follows.

Step 1. Initialize $\phi^{(0)}$ at the MLE (restricted to the interval $[\ln 0.02, \ln 50]$ in the simulation study).

Step 2. For $i = 1, \dots, M$, generate a candidate $\phi^* \sim N(\phi^{(i-1)}, s^2 \hat{\mathbf{V}})$, where $\hat{\mathbf{V}}$ is the inverse observed information of Section 3 used in **Equation 20**, and the scalar s is tuned every 50 iterations during burn-in toward an acceptance rate of about 25% and then kept fixed.

Step 3. Compute the acceptance probability, with the target density of **Equation 26**,

$$\rho = \min\left\{1, \frac{\pi(\phi^* \mid \text{data})}{\pi(\phi^{(i-1)} \mid \text{data})}\right\}. \quad (27)$$

Step 4. Draw $u \sim U(0, 1)$; with ρ from **Equation 27**, set $\phi^{(i)} = \phi^*$ if $u \leq \rho$, otherwise $\phi^{(i)} = \phi^{(i-1)}$.

Step 5. Discard the first l_B draws (burn-in), transform back to $(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}, \theta^{(i)})$, and compute $\delta^{(i)} = \alpha_1^{(i)} / (\alpha_1^{(i)} + \alpha_2^{(i)})$ for $i = l_B + 1, \dots, M$.

Step 6. Estimate δ by the Monte Carlo versions of **Equation 24** and **Equation 25**, as given in **Equation 28**,

$$\tilde{\delta}_{\text{SELF}} = \frac{1}{M - l_B} \sum_{i=l_B+1}^M \delta^{(i)}, \quad \tilde{\delta}_{\text{LINEX}} = -\frac{1}{c} \ln \left[\frac{1}{M - l_B} \sum_{i=l_B+1}^M e^{-c\delta^{(i)}} \right]. \quad (28)$$

A $100(1 - \gamma)\%$ highest posterior density (HPD) interval for δ is obtained from the sorted draws as the shortest interval containing $100(1 - \gamma)\%$ of them [34].

5. MONTE CARLO SIMULATION STUDY

5.1 Design

A Monte Carlo study with $B = 300$ replications per scenario is conducted to evaluate and compare the MLE and the Bayes estimators of δ . The simulation settings are as follows.

1. True parameter values: $(\alpha_1, \alpha_2) \in \{(0.6, 0.9), (0.6, 2.1), (1.8, 0.9), (1.8, 2.1)\}$, which give $\delta = 0.400, 0.222, 0.667$, and 0.462 by **Equation 6**, with $\beta = 1.5$ and $\theta = 1$.
2. Sample sizes $n = n_1 = n_2 \in \{40, 60\}$ and numbers of observed failures $m = m_1 = m_2$ equal to 50% and 75% of n , that is, $(n, m) \in \{(40, 20), (40, 30), (60, 30), (60, 45)\}$.
3. Removal probability $P = p_1 = p_2 \in \{0.05, 0.2, 0.5, 0.8\}$.
4. Scheme 1 (Sch-1), generated forward by **Equation 29**, and Scheme 2 (Sch-2), generated backward by **Equation 30**.
5. PT-IIC samples generated with the algorithm of Balakrishnan and Sandhu [35], using the quantile function in **Equation 31**.
6. MCMC with $M = 3000$ draws per replication, of which the first $l_B = 1000$ are discarded as burn-in. The informative priors are elicited as in Section 4, and the non-informative priors use $a_k = b_k = 10^{-3}$.
7. Loss functions: squared error, $\text{LINEX}_{-1.5}$, and $\text{LINEX}_{1.5}$.
8. Comparison measures: the average estimate and the RMSE in **Equation 32**, where $\tilde{\delta}_b$ is the estimate in replication b .

The two removal schemes are defined by

$$R_1 \sim \text{Bin}(n - m, P), \quad R_i \sim \text{Bin}\left(n - m - \sum_{s < i} R_s, P\right), \quad i = 2, \dots, m - 1, \quad R_m = n - m - \sum_{s < m} R_s, \quad (29)$$

$$R_m \sim \text{Bin}(n - m, P), \quad R_i \sim \text{Bin}\left(n - m - \sum_{s > i} R_s, P\right), \quad i = m - 1, \dots, 2, \quad R_1 = n - m - \sum_{s > 1} R_s. \quad (30)$$

Scheme 1 draws the removal counts sequentially from the first failure and assigns the remaining units to the last failure, whereas Scheme 2 starts from the last failure and assigns the remaining units to the first. For large P , Scheme 1 therefore removes most units early and Scheme 2 removes them late; for small P the roles are reversed. The quantile function used for sample generation and the performance measures are

$$F^{-1}(u) = \theta \left[(1 - u^{1/\alpha})^{-1/\beta} - 1 \right], \quad 0 < u < 1, \quad (31)$$

$$\text{Avg} = \frac{1}{B} \sum_{b=1}^B \tilde{\delta}_b, \quad \text{RMSE} = \sqrt{\frac{1}{B} \sum_{b=1}^B (\tilde{\delta}_b - \delta)^2}. \quad (32)$$

Equation 31 is the inverse of the CDF in **Equation 2**. All estimators are computed from the same simulated data in each replication. The Monte Carlo standard error of an RMSE is approximately $\text{RMSE}/\sqrt{2B}$, about 4% of its value, so differences of a few percent between neighboring rows of **Tables 1 to 4** are within simulation noise, and only systematic patterns should be interpreted.

5.2 Results

The results are reported in **Tables 1 to 4** for $\delta = 0.400, 0.222, 0.667$, and 0.462 , respectively. **Tables 1, 2, 3**, and **4** correspond to $(\alpha_1, \alpha_2) = (0.6, 0.9), (0.6, 2.1), (1.8, 0.9)$, and $(1.8, 2.1)$, respectively. In these tables, INF and N-INF denote the informative and non-informative priors, SELF denotes squared error loss, Sch. denotes the removal scheme, and the smallest RMSE in each row is shown in bold. **Table 5** summarizes the average RMSE over the scenarios of **Tables 1 to 4** within each group.

Table 1: Simulation results for $\delta = 0.400$

(n, m)	P	Sch.	MLE		Bayes: INF						Bayes: N-INF					
					SELF		LINEX _{-1.5}		LINEX _{1.5}		SELF		LINEX _{-1.5}		LINEX _{1.5}	
			Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE
(40, 20)	0.05	Sch-1	0.3982	0.0600	0.4034	0.0438	0.4052	0.0441	0.4015	0.0437	0.4013	0.0583	0.4037	0.0585	0.3989	0.0581
		Sch-2	0.3944	0.0582	0.3905	0.0427	0.3923	0.0424	0.3887	0.0430	0.3978	0.0558	0.4004	0.0559	0.3952	0.0559
	0.2	Sch-1	0.3937	0.0650	0.3977	0.0490	0.3998	0.0491	0.3956	0.0490	0.3975	0.0630	0.4002	0.0631	0.3949	0.0631
		Sch-2	0.3953	0.0581	0.4030	0.0439	0.4048	0.0442	0.4012	0.0438	0.3992	0.0557	0.4015	0.0558	0.3969	0.0557
	0.5	Sch-1	0.4011	0.0697	0.4013	0.0493	0.4034	0.0495	0.3992	0.0491	0.4037	0.0686	0.4067	0.0689	0.4008	0.0684
		Sch-2	0.4000	0.0593	0.4046	0.0453	0.4064	0.0455	0.4029	0.0450	0.4034	0.0575	0.4057	0.0578	0.4012	0.0573
	0.8	Sch-1	0.4041	0.0663	0.4008	0.0447	0.4029	0.0449	0.3986	0.0446	0.4072	0.0655	0.4104	0.0661	0.4041	0.0651
		Sch-2	0.3895	0.0582	0.3956	0.0439	0.3973	0.0439	0.3939	0.0440	0.3930	0.0559	0.3952	0.0558	0.3908	0.0561
(40, 30)	0.05	Sch-1	0.3971	0.0597	0.4006	0.0360	0.4020	0.0361	0.3992	0.0359	0.4004	0.0576	0.4027	0.0578	0.3982	0.0574
		Sch-2	0.4019	0.0561	0.4045	0.0394	0.4062	0.0397	0.4029	0.0392	0.4048	0.0551	0.4072	0.0554	0.4025	0.0548
	0.2	Sch-1	0.3978	0.0583	0.4027	0.0373	0.4043	0.0375	0.4011	0.0371	0.4007	0.0569	0.4031	0.0571	0.3983	0.0569
		Sch-2	0.3935	0.0560	0.3990	0.0354	0.4004	0.0355	0.3975	0.0354	0.3971	0.0544	0.3993	0.0545	0.3949	0.0545
	0.5	Sch-1	0.4019	0.0638	0.4026	0.0395	0.4042	0.0398	0.4010	0.0394	0.4042	0.0623	0.4067	0.0626	0.4016	0.0620
		Sch-2	0.4024	0.0605	0.4043	0.0461	0.4060	0.0464	0.4026	0.0459	0.4058	0.0580	0.4080	0.0584	0.4036	0.0577
	0.8	Sch-1	0.4005	0.0601	0.3968	0.0428	0.3986	0.0428	0.3950	0.0428	0.4034	0.0591	0.4060	0.0594	0.4008	0.0589
		Sch-2	0.3953	0.0592	0.3981	0.0465	0.3998	0.0466	0.3963	0.0465	0.3986	0.0576	0.4008	0.0577	0.3965	0.0576
(60, 30)	0.05	Sch-1	0.4021	0.0471	0.4065	0.0302	0.4076	0.0305	0.4054	0.0300	0.4048	0.0462	0.4064	0.0464	0.4032	0.0460
		Sch-2	0.3973	0.0494	0.3998	0.0341	0.4010	0.0342	0.3986	0.0341	0.4005	0.0482	0.4022	0.0483	0.3988	0.0481
	0.2	Sch-1	0.3977	0.0515	0.3987	0.0347	0.4000	0.0348	0.3974	0.0347	0.4003	0.0505	0.4022	0.0506	0.3985	0.0504
		Sch-2	0.3993	0.0452	0.3956	0.0298	0.3966	0.0297	0.3946	0.0299	0.4022	0.0445	0.4037	0.0446	0.4007	0.0443
	0.5	Sch-1	0.4033	0.0527	0.4012	0.0325	0.4026	0.0327	0.3999	0.0325	0.4054	0.0520	0.4075	0.0524	0.4034	0.0518
		Sch-2	0.3984	0.0461	0.4009	0.0360	0.4021	0.0361	0.3997	0.0360	0.4007	0.0451	0.4022	0.0452	0.3992	0.0451
	0.8	Sch-1	0.4005	0.0565	0.4002	0.0411	0.4019	0.0413	0.3986	0.0411	0.4024	0.0558	0.4046	0.0561	0.4001	0.0556
		Sch-2	0.3961	0.0448	0.3967	0.0311	0.3977	0.0311	0.3956	0.0312	0.3986	0.0441	0.4001	0.0441	0.3971	0.0441
(60, 45)	0.05	Sch-1	0.3990	0.0472	0.4011	0.0366	0.4024	0.0367	0.3999	0.0365	0.4019	0.0466	0.4034	0.0467	0.4003	0.0465
		Sch-2	0.4026	0.0431	0.3998	0.0309	0.4009	0.0309	0.3987	0.0308	0.4056	0.0425	0.4071	0.0428	0.4041	0.0423
	0.2	Sch-1	0.3933	0.0470	0.3953	0.0346	0.3965	0.0345	0.3940	0.0347	0.3949	0.0458	0.3966	0.0457	0.3933	0.0459
		Sch-2	0.4003	0.0441	0.4018	0.0305	0.4028	0.0306	0.4008	0.0304	0.4029	0.0432	0.4043	0.0434	0.4014	0.0430
	0.5	Sch-1	0.3955	0.0494	0.3936	0.0308	0.3947	0.0307	0.3926	0.0310	0.3972	0.0489	0.3990	0.0489	0.3955	0.0490
		Sch-2	0.3983	0.0470	0.3989	0.0362	0.4000	0.0362	0.3977	0.0362	0.4012	0.0457	0.4026	0.0458	0.3997	0.0456
	0.8	Sch-1	0.4018	0.0462	0.4008	0.0287	0.4019	0.0289	0.3996	0.0287	0.4039	0.0459	0.4057	0.0462	0.4021	0.0457
		Sch-2	0.4003	0.0448	0.3998	0.0266	0.4007	0.0266	0.3989	0.0266	0.4031	0.0443	0.4045	0.0444	0.4016	0.0441

The main findings from **Tables 1 to 5** are as follows.

1. *Effect of sample size and censoring fraction.* The RMSE of every estimator decreases as n and m grow. For the MLE, the average RMSE falls from 0.0590 at $(n, m) = (40, 20)$ to 0.0445 at $(60, 45)$. The average absolute bias of the average estimate does not exceed 0.004 for any estimator.
2. *Removal probability.* The effect of P is small. The average RMSE of the MLE increases slightly and monotonically, from 0.0505 at $P = 0.05$ to 0.0527 at $P = 0.8$ (about 4%), which is comparable to the Monte Carlo error of a single RMSE. By contrast, increasing m by 50% at fixed n lowers the average RMSE by about 6%, and increasing n from 40 to 60 lowers it by about 19%. The number of observed failures and the sample size are therefore the dominant factors.
3. *Removal scheme.* Scheme 2 yields a smaller average RMSE than Scheme 1 for the MLE (0.0497 versus 0.0536) and for the Bayes estimator with the non-informative prior (0.0484 versus 0.0528), whereas the two schemes are almost indistinguishable for the Bayes estimator with the informative prior (0.0393 versus 0.0399). The gap is concentrated at $P \geq 0.2$, where Scheme 1 removes units early. At $P = 0.05$, the average RMSE of the MLE is 0.0502 for Scheme 1 and 0.0509 for Scheme 2, and Scheme 2 is then the scheme with early removals. This pattern is consistent with the possibility that early removals reduce the effective information about the tail behavior of the lifetime distribution, although the present design does not permit a formal decomposition of the information loss. Expected Fisher information or a designed simulation varying the timing of removals could examine this issue.
4. *Bayes versus maximum likelihood.* The informative Bayes estimator under squared error loss has a smaller RMSE than the MLE and the non-informative Bayes estimator in all 128 scenarios. Its average RMSE is 0.0396, compared with 0.0516 for the MLE (23% smaller). The non-informative Bayes estimator improves on the MLE only slightly (average RMSE 0.0506, smaller than the MLE in 112 of 128 scenarios), as expected from a prior that carries almost no information. The advantage of the informative estimator should be read with care, because its prior is elicited from pilot samples generated under the true parameter values; it is therefore an optimistic upper bound on what can be gained when reliable prior information is available.

Table 2: Simulation results for $\delta = 0.222$

(n, m)	P	Sch.	MLE		Bayes: INF						Bayes: N-INF					
					SELF		LINEX _{-1.5}		LINEX _{1.5}		SELF		LINEX _{-1.5}		LINEX _{1.5}	
			Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE
(40, 20)	0.05	Sch-1	0.2173	0.0464	0.2285	0.0344	0.2299	0.0349	0.2272	0.0340	0.2283	0.0440	0.2301	0.0445	0.2266	0.0436
		Sch-2	0.2166	0.0509	0.2221	0.0398	0.2236	0.0401	0.2207	0.0396	0.2272	0.0487	0.2291	0.0492	0.2253	0.0482
	0.2	Sch-1	0.2154	0.0570	0.2212	0.0383	0.2225	0.0386	0.2198	0.0381	0.2254	0.0551	0.2275	0.0556	0.2235	0.0547
		Sch-2	0.2146	0.0485	0.2178	0.0376	0.2191	0.0377	0.2166	0.0376	0.2263	0.0469	0.2279	0.0473	0.2246	0.0466
	0.5	Sch-1	0.2205	0.0553	0.2271	0.0411	0.2287	0.0417	0.2255	0.0406	0.2288	0.0547	0.2311	0.0554	0.2267	0.0541
		Sch-2	0.2119	0.0457	0.2243	0.0330	0.2256	0.0333	0.2231	0.0328	0.2239	0.0423	0.2256	0.0426	0.2223	0.0420
	0.8	Sch-1	0.2163	0.0561	0.2208	0.0405	0.2224	0.0408	0.2192	0.0403	0.2245	0.0552	0.2267	0.0558	0.2222	0.0548
		Sch-2	0.2135	0.0491	0.2205	0.0353	0.2217	0.0355	0.2194	0.0352	0.2242	0.0468	0.2258	0.0472	0.2226	0.0465
	0.05	Sch-1	0.2198	0.0501	0.2276	0.0354	0.2287	0.0358	0.2265	0.0350	0.2286	0.0501	0.2302	0.0506	0.2270	0.0496
		Sch-2	0.2200	0.0446	0.2234	0.0325	0.2246	0.0328	0.2222	0.0323	0.2287	0.0445	0.2304	0.0450	0.2271	0.0440
(40, 30)	0.2	Sch-1	0.2210	0.0444	0.2238	0.0309	0.2250	0.0312	0.2226	0.0307	0.2281	0.0448	0.2299	0.0453	0.2264	0.0443
		Sch-2	0.2193	0.0447	0.2217	0.0353	0.2229	0.0356	0.2206	0.0352	0.2287	0.0441	0.2303	0.0445	0.2272	0.0437
	0.5	Sch-1	0.2117	0.0491	0.2174	0.0323	0.2186	0.0324	0.2162	0.0323	0.2190	0.0483	0.2208	0.0486	0.2173	0.0481
		Sch-2	0.2144	0.0430	0.2238	0.0320	0.2249	0.0322	0.2226	0.0318	0.2242	0.0416	0.2257	0.0419	0.2227	0.0413
	0.8	Sch-1	0.2232	0.0510	0.2298	0.0336	0.2310	0.0341	0.2287	0.0332	0.2299	0.0503	0.2317	0.0509	0.2281	0.0497
		Sch-2	0.2134	0.0478	0.2220	0.0382	0.2232	0.0385	0.2209	0.0380	0.2238	0.0456	0.2253	0.0459	0.2223	0.0453
(60, 30)	0.05	Sch-1	0.2206	0.0418	0.2277	0.0337	0.2287	0.0340	0.2268	0.0334	0.2298	0.0419	0.2311	0.0423	0.2286	0.0416
		Sch-2	0.2198	0.0435	0.2222	0.0308	0.2230	0.0310	0.2214	0.0307	0.2294	0.0426	0.2307	0.0430	0.2282	0.0422
	0.2	Sch-1	0.2180	0.0441	0.2257	0.0282	0.2265	0.0284	0.2248	0.0279	0.2249	0.0445	0.2263	0.0448	0.2235	0.0443
		Sch-2	0.2186	0.0375	0.2239	0.0257	0.2247	0.0259	0.2231	0.0256	0.2280	0.0368	0.2291	0.0371	0.2269	0.0365
	0.5	Sch-1	0.2224	0.0478	0.2295	0.0301	0.2304	0.0305	0.2286	0.0298	0.2282	0.0475	0.2297	0.0479	0.2267	0.0471
		Sch-2	0.2163	0.0385	0.2206	0.0287	0.2214	0.0287	0.2198	0.0286	0.2249	0.0365	0.2260	0.0367	0.2238	0.0363
	0.8	Sch-1	0.2211	0.0470	0.2229	0.0311	0.2239	0.0314	0.2219	0.0309	0.2263	0.0467	0.2278	0.0471	0.2247	0.0463
		Sch-2	0.2174	0.0392	0.2214	0.0316	0.2223	0.0317	0.2206	0.0315	0.2262	0.0384	0.2273	0.0387	0.2251	0.0381
(60, 45)	0.05	Sch-1	0.2217	0.0362	0.2244	0.0252	0.2252	0.0253	0.2237	0.0250	0.2277	0.0365	0.2288	0.0368	0.2266	0.0362
		Sch-2	0.2182	0.0368	0.2227	0.0244	0.2234	0.0245	0.2221	0.0243	0.2249	0.0370	0.2260	0.0372	0.2239	0.0368
	0.2	Sch-1	0.2212	0.0410	0.2217	0.0291	0.2226	0.0292	0.2209	0.0290	0.2259	0.0413	0.2271	0.0416	0.2247	0.0410
		Sch-2	0.2230	0.0373	0.2270	0.0276	0.2277	0.0279	0.2262	0.0274	0.2301	0.0376	0.2312	0.0379	0.2291	0.0372
	0.5	Sch-1	0.2220	0.0411	0.2237	0.0310	0.2246	0.0312	0.2229	0.0309	0.2265	0.0410	0.2277	0.0413	0.2253	0.0407
		Sch-2	0.2228	0.0346	0.2264	0.0255	0.2271	0.0257	0.2256	0.0253	0.2302	0.0355	0.2312	0.0358	0.2292	0.0351
	0.8	Sch-1	0.2196	0.0393	0.2227	0.0256	0.2235	0.0258	0.2220	0.0255	0.2238	0.0393	0.2250	0.0395	0.2226	0.0391
		Sch-2	0.2212	0.0333	0.2225	0.0255	0.2233	0.0256	0.2218	0.0254	0.2287	0.0335	0.2297	0.0339	0.2277	0.0332

5. *LINEX loss.* In all 128 scenarios, for both priors, the estimates satisfy $\text{LINEX}_{-1.5} > \text{squared error} > \text{LINEX}_{1.5}$, as implied by Jensen's inequality applied to **Equation 25**. The differences between the two LINEX estimates are small (0.0030 on average for the informative prior and 0.0038 for the non-informative prior) because the posterior of δ is concentrated, and the three loss functions have practically the same RMSE.
6. *Effect of the true value of the reliability.* The average RMSE of the MLE is smallest for $\delta = 0.222$ (0.0445) and largest for $\delta = 0.462$ (0.0562), with 0.0517 for $\delta = 0.667$ and 0.0541 for $\delta = 0.400$. This ordering follows that of $\delta(1 - \delta)$, which equals 0.173, 0.222, 0.240, and 0.249 for these four values and governs the sampling variability of a proportion-type parameter. The non-informative Bayes estimator shows the same ordering.

Two limitations of the simulation should be noted. First, convergence diagnostics are reported only for the real-data analysis (Section 6); no replication-level screening of the short chains ($M = 3000$, 2000 retained draws) is reported, so Monte Carlo error of the sampler may contribute to the small differences among the Bayes estimators. Second, the informative priors use information that would not be available in practice, as discussed above. Third, only point estimators are evaluated; the coverage probability and length of the asymptotic and highest posterior density intervals were not examined and merit a separate study.

6. APPLICATION TO REAL DATA

6.1 Data and Model Adequacy

The classical insulating-fluid data of Nelson [36] are analyzed. They consist of the times, in minutes, to breakdown of an insulating fluid between electrodes at constant voltages of 34 kV ($n = 19$) and 36 kV ($n = 15$), listed in **Table 6**. The probability that the fluid survives longer at 34 kV than at 36 kV is a natural SS quantity: the breakdown time at 34 kV plays the role of the strength V , and the breakdown time at 36 kV, which is shorter on average because the electrical stress is

Table 3: Simulation results for $\delta = 0.667$

(n, m)	P	Sch.	MLE		Bayes: INF						Bayes: N-INF					
					SELF		LINEX _{-1.5}		LINEX _{1.5}		SELF		LINEX _{-1.5}		LINEX _{1.5}	
			Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE
(40, 20)	0.05	Sch-1	0.6740	0.0572	0.6669	0.0423	0.6685	0.0422	0.6652	0.0425	0.6679	0.0547	0.6700	0.0546	0.6657	0.0549
		Sch-2	0.6711	0.0624	0.6639	0.0515	0.6660	0.0512	0.6618	0.0519	0.6650	0.0594	0.6674	0.0592	0.6626	0.0598
	0.2	Sch-1	0.6700	0.0580	0.6639	0.0425	0.6658	0.0422	0.6620	0.0429	0.6649	0.0567	0.6675	0.0566	0.6624	0.0570
		Sch-2	0.6689	0.0520	0.6647	0.0418	0.6665	0.0416	0.6630	0.0420	0.6626	0.0502	0.6647	0.0499	0.6605	0.0505
	0.5	Sch-1	0.6737	0.0711	0.6668	0.0489	0.6688	0.0487	0.6647	0.0492	0.6700	0.0690	0.6728	0.0690	0.6672	0.0692
		Sch-2	0.6751	0.0547	0.6695	0.0446	0.6713	0.0445	0.6678	0.0447	0.6693	0.0518	0.6713	0.0517	0.6672	0.0519
	0.8	Sch-1	0.6773	0.0640	0.6691	0.0472	0.6714	0.0471	0.6668	0.0473	0.6743	0.0625	0.6772	0.0627	0.6713	0.0625
		Sch-2	0.6713	0.0568	0.6664	0.0465	0.6682	0.0464	0.6647	0.0467	0.6659	0.0544	0.6679	0.0542	0.6638	0.0547
	0.05	Sch-1	0.6719	0.0541	0.6644	0.0383	0.6659	0.0381	0.6629	0.0386	0.6660	0.0529	0.6681	0.0527	0.6639	0.0531
		Sch-2	0.6670	0.0547	0.6612	0.0395	0.6628	0.0392	0.6595	0.0399	0.6623	0.0540	0.6644	0.0537	0.6601	0.0544
(40, 30)	0.2	Sch-1	0.6773	0.0559	0.6713	0.0420	0.6730	0.0421	0.6696	0.0421	0.6730	0.0538	0.6752	0.0539	0.6708	0.0538
		Sch-2	0.6732	0.0547	0.6680	0.0449	0.6697	0.0448	0.6664	0.0451	0.6678	0.0529	0.6698	0.0528	0.6658	0.0530
	0.5	Sch-1	0.6706	0.0554	0.6677	0.0437	0.6695	0.0435	0.6658	0.0439	0.6678	0.0553	0.6701	0.0551	0.6654	0.0555
		Sch-2	0.6713	0.0529	0.6655	0.0459	0.6673	0.0457	0.6637	0.0462	0.6662	0.0517	0.6681	0.0516	0.6642	0.0520
	0.8	Sch-1	0.6700	0.0591	0.6698	0.0478	0.6718	0.0478	0.6679	0.0479	0.6661	0.0588	0.6685	0.0586	0.6637	0.0591
		Sch-2	0.6739	0.0553	0.6675	0.0441	0.6691	0.0440	0.6658	0.0443	0.6684	0.0535	0.6704	0.0534	0.6665	0.0536
(60, 30)	0.05	Sch-1	0.6694	0.0437	0.6639	0.0311	0.6650	0.0309	0.6628	0.0313	0.6638	0.0429	0.6654	0.0427	0.6623	0.0431
		Sch-2	0.6729	0.0453	0.6707	0.0365	0.6719	0.0366	0.6694	0.0366	0.6680	0.0437	0.6696	0.0437	0.6665	0.0438
	0.2	Sch-1	0.6680	0.0501	0.6658	0.0389	0.6672	0.0388	0.6644	0.0391	0.6641	0.0499	0.6659	0.0497	0.6623	0.0501
		Sch-2	0.6752	0.0478	0.6690	0.0401	0.6702	0.0401	0.6678	0.0401	0.6710	0.0457	0.6723	0.0458	0.6696	0.0457
	0.5	Sch-1	0.6684	0.0486	0.6618	0.0357	0.6633	0.0354	0.6603	0.0361	0.6658	0.0485	0.6677	0.0484	0.6638	0.0487
		Sch-2	0.6697	0.0450	0.6675	0.0351	0.6686	0.0350	0.6663	0.0351	0.6653	0.0432	0.6667	0.0431	0.6639	0.0434
	0.8	Sch-1	0.6650	0.0498	0.6624	0.0334	0.6638	0.0331	0.6610	0.0337	0.6625	0.0489	0.6645	0.0486	0.6603	0.0492
		Sch-2	0.6690	0.0419	0.6659	0.0324	0.6670	0.0323	0.6648	0.0325	0.6648	0.0406	0.6662	0.0404	0.6634	0.0407
(60, 45)	0.05	Sch-1	0.6696	0.0450	0.6676	0.0380	0.6688	0.0380	0.6664	0.0381	0.6659	0.0444	0.6673	0.0443	0.6644	0.0445
		Sch-2	0.6672	0.0448	0.6636	0.0325	0.6646	0.0323	0.6625	0.0327	0.6638	0.0436	0.6652	0.0434	0.6624	0.0438
	0.2	Sch-1	0.6699	0.0460	0.6690	0.0363	0.6702	0.0363	0.6679	0.0363	0.6676	0.0457	0.6691	0.0457	0.6661	0.0458
		Sch-2	0.6676	0.0415	0.6690	0.0337	0.6701	0.0337	0.6679	0.0337	0.6633	0.0412	0.6647	0.0410	0.6619	0.0414
	0.5	Sch-1	0.6679	0.0472	0.6672	0.0369	0.6685	0.0368	0.6660	0.0370	0.6650	0.0468	0.6666	0.0466	0.6634	0.0470
		Sch-2	0.6713	0.0467	0.6699	0.0388	0.6710	0.0389	0.6688	0.0389	0.6670	0.0457	0.6683	0.0457	0.6657	0.0458
	0.8	Sch-1	0.6672	0.0465	0.6619	0.0325	0.6630	0.0322	0.6607	0.0327	0.6652	0.0463	0.6668	0.0462	0.6635	0.0465
		Sch-2	0.6686	0.0459	0.6663	0.0361	0.6673	0.0360	0.6652	0.0362	0.6639	0.0446	0.6653	0.0445	0.6626	0.0448

higher, plays the role of the stress W . The Shapiro-Wilk test rejects normality for both data sets ($p = 9.7 \times 10^{-5}$ for V and $p = 3.0 \times 10^{-5}$ for W), in line with the right-skewed, heavy-tailed shape of the data.

The EL distribution is compared with three competing models, whose PDFs are given in **Table 7**, by means of the Akaike and Bayesian information criteria (AIC and BIC) [37, 38], the Kolmogorov-Smirnov statistic (K-ST), and the Anderson-Darling statistic (ADT). Because the parameters are estimated, the p -values of K-ST and ADT are computed by a parametric bootstrap with 500 replications. The fitted parameters and goodness-of-fit measures are reported in **Table 8**, where NLC denotes the negative log-likelihood, the best value of each criterion is shown in bold, and, for the Weibull model, $\hat{\beta}$ and $\hat{\theta}$ are the shape and scale estimates, whereas for the exponential model $\hat{\theta}$ is the mean.

The EL distribution is not rejected for either data set (bootstrap p -values of K-ST: 0.180 for V and 0.826 for W), and its fitted CDF follows the empirical CDF closely, as shown in **Figure 3**. For W , the EL model has the smallest AIC (77.46) and the largest p -values of K-ST and ADT, whereas the exponential model has the smallest BIC (78.53). For V , the log-likelihoods of the EL, Lomax, and Weibull models are practically identical (NLC = 68.41, 68.42, and 68.39), so the information criteria, which penalize the additional parameter of the EL model, slightly favor the Lomax and Weibull models (AIC 140.85 and 140.77 versus 142.83), and the exponential model has the smallest BIC (142.19). The exponential model is rejected at the 5% level by K-ST ($p = 0.026$) and ADT ($p = 0.042$) for V . In summary, the EL distribution is an adequate and competitive model for both data sets, although it is not uniformly the best according to the information criteria.

The SS formula in **Equation 6** requires that the two populations share β and θ . This was tested with the likelihood ratio test of $H_0 : (\beta_V, \theta_V) = (\beta_W, \theta_W)$ against the alternative of distinct parameters, using the complete samples. The test statistic is 4.835 on 2 degrees of freedom ($p = 0.089$), so H_0 is not rejected at the 5% level. This is weak support for the common-parameter model rather than strong confirmation, because the test has limited power with samples of 19 and 15 observations. Under H_0 , the complete-sample MLE is $\hat{\delta} = 0.601$ (95% ACI 0.433 to 0.749). The model-free estimate, namely the proportion of the 19×15 pairs with $w < v$, with the single tie counted as one half, is 0.714. The two estimates differ by 0.113. The model-free estimate has a bootstrap standard error of about 0.09, so the difference is about 1.2 standard errors and is compatible with sampling variability. It may also reflect the smoothing imposed by the parametric common-parameter model, which the likelihood ratio test does not clearly support.

Table 4: Simulation results for $\delta = 0.462$

(n, m)	P	Sch.	MLE		Bayes: INF						Bayes: N-INF					
					SELF		LINEX _{-1.5}		LINEX _{1.5}		SELF		LINEX _{-1.5}		LINEX _{1.5}	
			Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE	Avg.	RMSE
(40, 20)	0.05	Sch-1	0.4534	0.0645	0.4558	0.0574	0.4580	0.0573	0.4535	0.0577	0.4551	0.0624	0.4575	0.0623	0.4527	0.0626
		Sch-2	0.4510	0.0640	0.4528	0.0531	0.4551	0.0529	0.4505	0.0535	0.4531	0.0617	0.4558	0.0614	0.4504	0.0621
	0.2	Sch-1	0.4653	0.0641	0.4682	0.0570	0.4708	0.0574	0.4656	0.0566	0.4657	0.0629	0.4685	0.0631	0.4629	0.0627
		Sch-2	0.4576	0.0574	0.4600	0.0526	0.4622	0.0526	0.4578	0.0526	0.4588	0.0558	0.4611	0.0558	0.4565	0.0560
	0.5	Sch-1	0.4524	0.0693	0.4527	0.0579	0.4554	0.0576	0.4500	0.0583	0.4529	0.0680	0.4561	0.0678	0.4498	0.0684
		Sch-2	0.4538	0.0588	0.4569	0.0517	0.4590	0.0516	0.4548	0.0519	0.4551	0.0572	0.4574	0.0571	0.4528	0.0574
	0.8	Sch-1	0.4572	0.0718	0.4606	0.0585	0.4636	0.0586	0.4577	0.0585	0.4578	0.0705	0.4612	0.0705	0.4545	0.0707
		Sch-2	0.4616	0.0590	0.4631	0.0542	0.4653	0.0543	0.4608	0.0542	0.4620	0.0577	0.4642	0.0577	0.4597	0.0577
(40, 30)	0.05	Sch-1	0.4616	0.0583	0.4635	0.0514	0.4656	0.0515	0.4614	0.0513	0.4621	0.0566	0.4645	0.0567	0.4598	0.0566
		Sch-2	0.4611	0.0576	0.4626	0.0510	0.4648	0.0511	0.4604	0.0509	0.4618	0.0573	0.4642	0.0574	0.4594	0.0573
	0.2	Sch-1	0.4618	0.0595	0.4624	0.0495	0.4646	0.0496	0.4602	0.0495	0.4627	0.0597	0.4652	0.0598	0.4602	0.0596
		Sch-2	0.4581	0.0596	0.4597	0.0525	0.4618	0.0525	0.4576	0.0526	0.4596	0.0585	0.4619	0.0585	0.4573	0.0586
	0.5	Sch-1	0.4617	0.0626	0.4616	0.0527	0.4638	0.0528	0.4594	0.0526	0.4621	0.0624	0.4648	0.0626	0.4595	0.0624
		Sch-2	0.4665	0.0574	0.4672	0.0492	0.4692	0.0496	0.4651	0.0490	0.4680	0.0563	0.4703	0.0567	0.4657	0.0561
	0.8	Sch-1	0.4662	0.0653	0.4659	0.0575	0.4683	0.0578	0.4635	0.0572	0.4665	0.0652	0.4692	0.0655	0.4638	0.0649
		Sch-2	0.4618	0.0591	0.4625	0.0511	0.4645	0.0512	0.4605	0.0511	0.4634	0.0575	0.4656	0.0577	0.4612	0.0575
(60, 30)	0.05	Sch-1	0.4619	0.0495	0.4624	0.0405	0.4639	0.0406	0.4609	0.0405	0.4628	0.0486	0.4645	0.0486	0.4611	0.0485
		Sch-2	0.4630	0.0519	0.4638	0.0465	0.4654	0.0466	0.4622	0.0464	0.4639	0.0514	0.4657	0.0516	0.4622	0.0513
	0.2	Sch-1	0.4665	0.0539	0.4646	0.0460	0.4663	0.0462	0.4629	0.0458	0.4669	0.0535	0.4689	0.0538	0.4650	0.0533
		Sch-2	0.4576	0.0502	0.4579	0.0450	0.4594	0.0449	0.4565	0.0451	0.4590	0.0493	0.4605	0.0493	0.4574	0.0494
	0.5	Sch-1	0.4604	0.0557	0.4635	0.0452	0.4653	0.0453	0.4616	0.0451	0.4608	0.0561	0.4630	0.0561	0.4586	0.0561
		Sch-2	0.4576	0.0485	0.4600	0.0435	0.4614	0.0435	0.4586	0.0435	0.4589	0.0482	0.4604	0.0482	0.4573	0.0483
	0.8	Sch-1	0.4627	0.0566	0.4633	0.0464	0.4653	0.0466	0.4613	0.0464	0.4624	0.0571	0.4648	0.0573	0.4600	0.0571
		Sch-2	0.4607	0.0514	0.4615	0.0468	0.4629	0.0468	0.4601	0.0468	0.4614	0.0499	0.4629	0.0500	0.4599	0.0499
(60, 45)	0.05	Sch-1	0.4604	0.0418	0.4614	0.0364	0.4629	0.0364	0.4600	0.0364	0.4606	0.0413	0.4622	0.0413	0.4590	0.0413
		Sch-2	0.4625	0.0511	0.4632	0.0455	0.4646	0.0456	0.4618	0.0455	0.4635	0.0508	0.4651	0.0510	0.4619	0.0508
	0.2	Sch-1	0.4615	0.0502	0.4611	0.0432	0.4626	0.0432	0.4596	0.0432	0.4619	0.0502	0.4637	0.0503	0.4602	0.0502
		Sch-2	0.4583	0.0481	0.4607	0.0419	0.4621	0.0419	0.4593	0.0419	0.4595	0.0475	0.4611	0.0475	0.4580	0.0475
	0.5	Sch-1	0.4634	0.0526	0.4630	0.0435	0.4645	0.0436	0.4614	0.0434	0.4638	0.0525	0.4657	0.0526	0.4620	0.0524
		Sch-2	0.4589	0.0465	0.4612	0.0398	0.4625	0.0398	0.4598	0.0398	0.4598	0.0459	0.4613	0.0459	0.4583	0.0460
	0.8	Sch-1	0.4587	0.0535	0.4649	0.0434	0.4665	0.0436	0.4633	0.0432	0.4592	0.0536	0.4610	0.0536	0.4573	0.0537
		Sch-2	0.4593	0.0496	0.4600	0.0434	0.4614	0.0434	0.4587	0.0435	0.4599	0.0487	0.4615	0.0487	0.4584	0.0488

Table 5: Average RMSE of the estimators of δ by scenario group

Group	MLE		Bayes: INF			Bayes: N-INF		
			SELF	LINEX _{-1.5}	LINEX _{1.5}	SELF	LINEX _{-1.5}	LINEX _{1.5}
All scenarios		0.0516	0.0396	0.0397	0.0396	0.0506	0.0507	0.0505
$(n, m) = (40, 20)$		0.0590	0.0459	0.0460	0.0460	0.0572	0.0573	0.0571
$(n, m) = (40, 30)$		0.0553	0.0423	0.0424	0.0423	0.0543	0.0544	0.0542
$(n, m) = (60, 30)$		0.0476	0.0360	0.0361	0.0360	0.0468	0.0470	0.0468
$(n, m) = (60, 45)$		0.0445	0.0341	0.0341	0.0341	0.0442	0.0443	0.0441
$P = 0.05$		0.0505	0.0388	0.0389	0.0388	0.0494	0.0495	0.0493
$P = 0.2$		0.0509	0.0392	0.0393	0.0392	0.0500	0.0501	0.0499
$P = 0.5$		0.0524	0.0401	0.0402	0.0401	0.0514	0.0515	0.0513
$P = 0.8$		0.0526	0.0403	0.0403	0.0402	0.0517	0.0518	0.0516
Sch-1		0.0536	0.0399	0.0400	0.0399	0.0528	0.0530	0.0528
Sch-2		0.0497	0.0393	0.0393	0.0393	0.0484	0.0485	0.0483
$\delta = 0.400$		0.0541	0.0378	0.0379	0.0378	0.0528	0.0530	0.0527
$\delta = 0.222$		0.0445	0.0320	0.0323	0.0318	0.0437	0.0441	0.0434
$\delta = 0.667$		0.0517	0.0400	0.0399	0.0402	0.0504	0.0503	0.0506
$\delta = 0.462$		0.0562	0.0486	0.0486	0.0486	0.0554	0.0555	0.0555

Table 6: Breakdown times of an insulating fluid (minutes) at 34 kV and 36 kV

Data set	Observations (minutes)
V (34 kV, $n = 19$)	0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89
W (36 kV, $n = 15$)	0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50

Table 7: Competing distributions used in the goodness-of-fit comparison

Distribution	PDF, $x > 0$	Parameters
EL	$f(x) = \frac{\alpha\beta}{\theta} \left(1 + \frac{x}{\theta}\right)^{-\beta-1} \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\beta}\right]^{\alpha-1}$	$\alpha, \beta, \theta > 0$
Lomax	$f(x) = \frac{\beta}{\theta} \left(1 + \frac{x}{\theta}\right)^{-\beta-1}$	$\beta, \theta > 0$
Weibull	$f(x) = \frac{\beta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}$	β (shape), θ (scale) > 0
Exponential	$f(x) = \frac{1}{\theta} e^{-x/\theta}$	$\theta > 0$ (mean)

Table 8: Goodness-of-fit statistics for the two data sets

Data	Dist.	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	NLC	AIC	BIC	K-ST	p	ADT	p
V	EL	0.929	2.308	21.629	68.415	142.829	145.663	0.1472	0.180	0.329	0.308
	Lomax	–	2.032	16.748	68.423	140.847	142.736	0.1479	0.350	0.322	0.600
	Weibull	–	0.771	12.222	68.386	140.772	142.661	0.1613	0.208	0.392	0.386
	Exponential	–	–	14.359	69.623	141.246	142.191	0.2464	0.026	1.330	0.042
W	EL	3.289	1.587	1.248	35.728	77.456	79.580	0.1100	0.826	0.206	0.770
	Lomax	–	3.046	9.496	36.979	77.958	79.374	0.1435	0.604	0.454	0.486
	Weibull	–	0.889	4.292	37.691	79.383	80.799	0.1917	0.126	0.627	0.102
	Exponential	–	–	4.606	37.910	77.821	78.529	0.2205	0.148	0.803	0.150

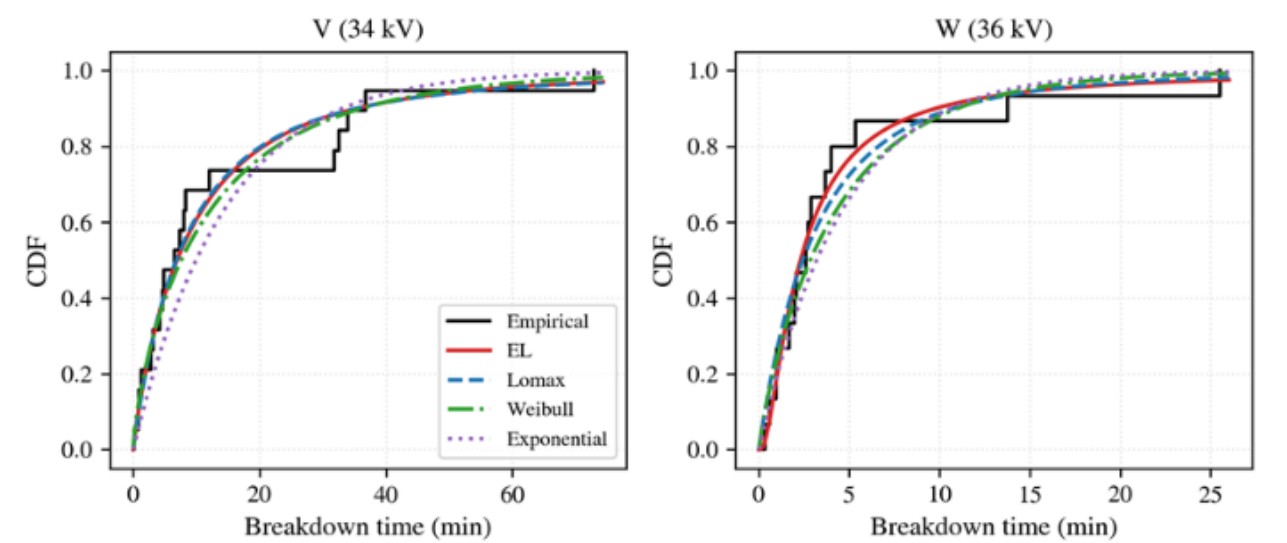


Figure 3: Empirical and fitted CDFs for data set V (34 kV, left) and data set W (36 kV, right)

6.2 Analysis of PT-IIC Samples with Binomial Removals

The original data are complete, not progressively censored. To illustrate the methodology, PT-IIC samples with $m_1 = m_2 = 10$ were therefore generated from the two data sets by simulated censoring under binomial removal with $p_1 = p_2 = 0.35$. The removal vectors were drawn from **Equation 16** and **Equation 17** (with $n_1 = 19$ and $n_2 = 15$), and at each observed failure the prescribed number of surviving units was withdrawn at random. The resulting samples and removal vectors (generated with a fixed random seed) are given in **Table 9**. The MLEs of the removal probabilities from **Equation 18** are $\hat{p}_1 = 0.474$ and $\hat{p}_2 = 0.278$. The maximum likelihood estimates of the lifetime parameters, obtained from **Equation 10** to **Equation 14**, and the delta-method standard errors are reported in **Table 10**, and $\hat{\delta}$ follows from **Equation 19**.

Table 9: PT-IIC samples and removal vectors

i	1	2	3	4	5	6	7	8	9	10
v_i	0.19	0.96	4.15	4.67	7.35	8.01	12.06	31.75	33.91	36.71
R_i	4	2	1	2	0	0	0	0	0	0
w_i	0.35	0.59	0.99	1.69	1.97	2.07	2.58	2.90	3.99	5.35
R_i°	1	2	0	0	0	1	1	0	0	0

Table 10: MLEs and standard errors from the PT-IIC samples

Parameter	MLE	Std. error
α_1	1.742	0.821
α_2	0.964	0.472
β	1.904	1.562
θ	8.220	12.382
δ	0.644	0.087

The Bayes estimates, computed under non-informative priors with the algorithm of Section 4, are given in **Table 11** together with the complete-sample MLE for comparison.

Table 11: Estimates of δ for the real data

Estimator	Estimate of δ	95% interval	Interval type
MLE	0.644	(0.462, 0.792)	ACI
Bayes, SELF	0.632	(0.462, 0.794)	HPD
Bayes, LINEX _{-1.5}	0.638	—	—
Bayes, LINEX _{1.5}	0.627	—	—
MLE, complete samples	0.601	(0.433, 0.749)	ACI

The MCMC results of **Table 11** are based on four independent chains of 60,000 iterations each, with the first 10,000 discarded, started from dispersed initial values. The potential scale reduction factor (PSRF) [39] of δ is 1.0001, and the total effective sample size is about 8,342. **Figure 4** shows the trace plots, the posterior density, and the cumulative means. The chains exhibit satisfactory mixing, the posterior of δ is unimodal and slightly left-skewed, and the cumulative means stabilize near 0.632. The posterior standard deviation is 0.086.

The SS reliability is estimated as $\hat{\delta} = 0.644$ by the MLE and as 0.632 by the Bayes estimator under squared error loss, so the probability that the fluid survives longer at 34 kV than at 36 kV is about 63%. The LINEX estimates (0.638 for $c = -1.5$ and 0.627 for $c = 1.5$) bracket the squared error estimate in the order predicted by theory. Both 95% intervals are wide (about 0.33 in length) and contain the complete-sample estimate 0.601 and the model-free estimate 0.714, which reflects the small number of observed failures ($m = 10$) in each sample. The longer-lived (34 kV) fluid is therefore more likely than not to outlast the 36 kV fluid, but the data provide only limited precision for δ .

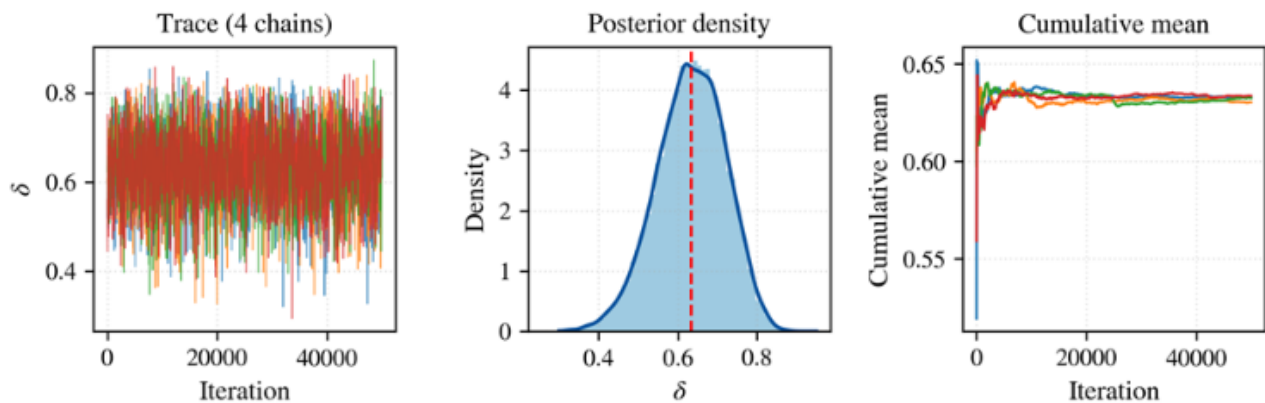


Figure 4: MCMC diagnostics for δ under the non-informative prior

7. CONCLUSION

This paper has studied the SS reliability $\delta = P(W < V)$ for the Exponentiated Lomax distribution under progressive Type-II censoring with binomial removals. For independent EL variables sharing β and θ , the reliability reduces to $\delta = \alpha_1 / (\alpha_1 + \alpha_2)$ in **Equation 6**, which depends only on the exponentiation parameters. For PT-IIC data with binomial removals, the likelihood in **Equation 7**, the analytic score equations in **Equation 10** to **Equation 14**, the MLE of δ in **Equation 19**, the closed-form estimators of the removal probabilities in **Equation 18**, and delta-method confidence intervals in **Equation 20** were obtained, and Bayes estimators under squared error and LINEX losses were computed by a Metropolis-Hastings sampler together with highest posterior density intervals. The Monte Carlo study of 128 scenarios with 300 replications each showed that the RMSEs of all estimators decrease as n and m increase and that the removal probability has only a small effect. The informative Bayes estimator had the smallest RMSE in all 128 scenarios (average 0.040 versus 0.052 for the MLE), whereas the non-informative Bayes estimator behaved similarly to the MLE (average 0.051); the benefit of the informative prior depends on the quality of the prior information, and in the simulation it is an optimistic upper bound because the priors were built from samples generated under the true parameters. For the insulating-fluid data, a PT-IIC sample with $m = 10$ per data set gave $\hat{\delta} = 0.644$ and a Bayes estimate of 0.632 under squared error loss, with wide intervals because of the small number of failures. Two limitations should be noted. The closed-form expression relies on the common-parameter assumption, which was not rejected for the real data (likelihood ratio test, $p = 0.089$) but is only weakly supported, and the case of different β and θ requires numerical integration. In addition, the likelihood of the EL model can be flat for small samples, which motivates weakly informative priors and larger numbers of observed failures, and the coverage of the interval estimators was not evaluated. Future research could extend the proposed methods to other censoring schemes, such as adaptive progressive or hybrid censoring.

AUTHOR CONTRIBUTION STATEMENT

The author contributed to the study conception and design, material preparation, data collection, and analysis. The author read and approved the final manuscript.

ETHICS APPROVAL AND CONSENT TO PARTICIPATE

Ethics declaration: not applicable. This study did not involve human participants or animals.

CONSENT FOR PUBLICATION

Not Applicable.

DATA AVAILABILITY

All data generated or analyzed during this study are included in this published article.

ACKNOWLEDGMENTS

The author acknowledges the use of an AI-based assistant for support with code development and statistical re-analysis. The entire manuscript remains the sole responsibility of the author.

FUNDING

No Funding.

DISCLOSURE STATEMENT

The author declares no conflict of interest.

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