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# Chebyshev-Gauss Collocation with Exact Fractional Integration for Multiterm Variable-Order Fractional Differential Equations

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## ABSTRACT

A collocation method is developed for multiterm variable-order fractional differential equations with Caputo derivatives. The highest integer-order derivative of the solution is expanded in shifted Chebyshev polynomials of the first kind, and the solution and all of its variable-order derivatives are recovered by exact integration of the monomial form of the basis. No operational matrix is formed, and the initial conditions hold by construction. Enforcing the equation at the Chebyshev-Gauss points yields a square algebraic system, which is linear for linear equations and is otherwise solved by Newton's method. The variable-order integral operator is bounded on  $L^2(0, 1)$  with an explicit constant, and solutions that are polynomials of degree at most  $N + l$  are reproduced exactly, where  $N$  is the truncation degree and  $l$  is the number of initial conditions. An error estimate of order  $N^{1/2-s}$  is derived in terms of the weighted Sobolev regularity  $s$  of the  $l$ th derivative; it is conditional on a discrete stability hypothesis that is stated but not proved. Six linear benchmark problems with polynomial solutions, including the Bagley-Torvik equation and a problem with the non-polynomial order  $\alpha(x) = (x + 2e^x)/7$ , are solved with at most four basis functions, with maximum errors from  $5.55 \times 10^{-17}$  to  $4.44 \times 10^{-14}$ . These values are at the rounding level, verify the implementation, and are of the same order as published results for operational matrix and hybrid block-pulse schemes, so no accuracy advantage is claimed. Two further linear tests with non-polynomial solutions show exponential convergence for a smooth solution and an observed algebraic rate of about  $N^{-3.5}$  for a solution of limited regularity. The monomial representation restricts double precision arithmetic to roughly  $N \leq 24$  and  $N \leq 18$  in these two tests, respectively. The nonlinear branch of the algorithm is not tested numerically.

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## 1. INTRODUCTION

Fractional calculus provides a standard framework for modeling systems with memory and hereditary effects. Constant-order models have been treated by a wide range of spectral schemes [1, 2, 3, 4, 5]. In many physical situations, however, the order of differentiation itself varies with time, which led to the introduction of variable-order operators [6, 7]. Reported applications include nonlinear variable-order oscillators [8], creep of rock [9], and fractional-variable-order controllers [10].

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Closed-form solutions of variable-order fractional differential equations (VOFDEs) are rarely available, so numerical methods are required. For variable-order equations, Legendre wavelets [11], an operational matrix of differentiation based on second-kind Chebyshev polynomials [12], an operational matrix based on fourth-kind Chebyshev polynomials [13], and operational matrices based on shifted Legendre polynomials [14] have been used. An operational matrix of integration based on shifted second-kind Chebyshev polynomials was developed earlier for fractional equations of constant order [15]. Hybrid functions that combine block-pulse functions with Bernoulli or Fibonacci polynomials have also been applied to fractional problems [16, 17], and Bernoulli polynomials have been used for variable-order equations [18]. In the operational matrix schemes, the variable-order derivative of the basis vector is written as a matrix that depends on  $x$  [12, 14], and the monomial rule of Lemma 1 underlies the evaluation of such matrices for polynomial bases.

The present work follows the same exact-integration principle for a global basis of shifted Chebyshev polynomials; the principle itself is not claimed as new. The highest integer-order derivative of the solution is approximated by a truncated Chebyshev series, and the variable-order integral of every basis function is evaluated in closed form from the monomial expansion. Collocation at the Chebyshev-Gauss points then leads to a square algebraic system, and the resulting scheme is referred to as the Chebyshev-Gauss collocation method (CGCM). Compared with the operational matrix schemes, the formulation differs in three respects. The initial conditions hold identically, no change-of-basis matrix is inverted, and the nodes are the Chebyshev-Gauss points, for which the discrete norm used in Section 4 is exact. These points lie in the open interval  $(0, 1)$ , so the residual is never evaluated at  $x = 0$ , where powers of the form  $x^{m-\gamma(x)}$  can be indeterminate.

The scheme has a practical restriction. The monomial form of the basis loses accuracy by cancellation as  $N$  grows, so double precision arithmetic is reliable only for moderate  $N$ ; this restriction is quantified in Section 5.5. In addition, all numerical tests are linear, so the nonlinear branch of the algorithm is formulated but not verified numerically. The six benchmark solutions of Section 5 are polynomials lying in the span of the basis, so the corresponding results verify the implementation and do not by themselves establish superiority over other schemes. Two additional tests with non-polynomial solutions are therefore included to examine convergence in  $N$  and the conditioning of the monomial form. The contributions are as follows.

1. A closed-form expression for the variable-order Riemann-Liouville integral of shifted Chebyshev polynomials, and the resulting collocation scheme for multiterm Caputo VOFDEs.
2. An  $L^2$  bound for the variable-order integral operator with an explicit constant, a proof that polynomial solutions of degree at most  $N + l$  are reproduced exactly, and an error estimate that holds under a stated discrete stability hypothesis, which is not proved.
3. A step-by-step algorithm, including the treatment of initial conditions and an analytic Jacobian for Newton's iteration.
4. Numerical results for six linear benchmark problems with explicit source terms and a comparison against published values, together with two non-polynomial tests that quantify the convergence rate and the limits of double precision arithmetic.

The remainder of the paper is organized as follows. Section 2 collects the definitions and lemmas. Section 3 describes the collocation scheme and **Algorithm 1**. Section 4 contains the analysis. Section 5 reports the numerical experiments, and Section 6 gives the conclusions.

## 2. PRELIMINARIES

### 2.1 Variable-Order Fractional Calculus

Throughout,  $l \in \mathbb{N}$  denotes the number of initial conditions, and every variable order  $\gamma$  is a measurable function on  $(0, 1]$  with  $0 \leq \gamma(x) \leq l$ . The integer  $\lceil \gamma(x) \rceil$  is evaluated pointwise.

**Definition 1** (Caputo derivative). For  $f \in C^l[0, 1]$  and a variable order  $\gamma(x) \geq 0$ , the left Caputo derivative is [19]

$$D^{\gamma(x)} f(x) = \begin{cases} \frac{1}{\Gamma(n - \gamma(x))} \int_0^x (x - s)^{n - \gamma(x) - 1} f^{(n)}(s) ds, & n - 1 < \gamma(x) < n, \\ f^{(n)}(x), & \gamma(x) = n, \end{cases} \quad (1)$$

where  $n = \lceil \gamma(x) \rceil \in \mathbb{N}$  and  $\Gamma(\cdot)$  is the gamma function. For  $\gamma(x) = 0$ ,  $D^0 f = f$ .

**Definition 2** (Riemann-Liouville integral). For a variable order  $\gamma(x) \geq 0$ , the left Riemann-Liouville integral is [19]

$$I^{\gamma(x)} f(x) = \frac{1}{\Gamma(\gamma(x))} \int_0^x (x - s)^{\gamma(x) - 1} f(s) ds, \quad x > 0, \quad (2)$$

and  $I^0 f = f$ .

In both definitions the order is evaluated at the outer point  $x$ . This is the convention of [11, 12, 14], without the initialization term that appears in the first definition of [12, 14]. Consequently, for every fixed  $x$  the operators coincide with constant-order operators of order  $\gamma(x)$ , and the semigroup property  $I^a I^m g(x) = I^{a+m} g(x)$  holds pointwise for a constant  $a \geq 0$  and an integer  $m \geq 1$ .

**Lemma 1.** For  $\mu > -1$  and  $\gamma(x) \geq 0$ ,

$$I^{\gamma(x)} x^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + \gamma(x))} x^{\mu + \gamma(x)}. \quad (3)$$

*Proof.* For  $\gamma(x) > 0$ , the substitution  $s = xt$  gives  $\Gamma(\gamma(x))^{-1} x^{\mu + \gamma(x)} \int_0^1 (1-t)^{\gamma(x)-1} t^\mu dt$ . The integral is the beta function  $B(\mu + 1, \gamma(x)) = \Gamma(\mu + 1)\Gamma(\gamma(x))/\Gamma(\mu + 1 + \gamma(x))$ . The case  $\gamma(x) = 0$  is trivial.  $\square$

**Lemma 2.** Let  $l \in \mathbb{N}$  satisfy  $l \geq \lceil \gamma(x) \rceil$  for all  $x \in (0, 1]$ , and let  $f \in C^l[0, 1]$ . Then

$$D^{\gamma(x)} f(x) = I^{l-\gamma(x)} f^{(l)}(x) + \sum_{k=\lceil \gamma(x) \rceil}^{l-1} \frac{f^{(k)}(0)}{\Gamma(k+1-\gamma(x))} x^{k-\gamma(x)}, \quad x \in (0, 1], \quad (4)$$

where the sum is empty when  $\lceil \gamma(x) \rceil = l$ .

*Proof.* Fix  $x$  and put  $n = \lceil \gamma(x) \rceil \leq l$ . By Definition 1,  $D^{\gamma(x)} f(x) = I^{n-\gamma(x)} f^{(n)}(x)$ . Taylor's formula with integral remainder gives, for  $s \in [0, x]$ ,

$$f^{(n)}(s) = \sum_{k=n}^{l-1} \frac{f^{(k)}(0)}{(k-n)!} s^{k-n} + I^{l-n} f^{(l)}(s).$$

Applying  $I^{n-\gamma(x)}$  with the order frozen at  $x$ , Lemma 1 converts each monomial to  $f^{(k)}(0)x^{k-\gamma(x)}/\Gamma(k+1-\gamma(x))$ , and the semigroup property turns  $I^{n-\gamma(x)} I^{l-n} f^{(l)}$  into  $I^{l-\gamma(x)} f^{(l)}$ .  $\square$

The statement admits any integer  $l \geq \lceil \gamma(x) \rceil$ , so a single integer  $l$  serves all terms of a multiterm equation.

**Remark 1** (Crossing of an integer order). Let  $\gamma(x)$  cross an integer  $n$  at a point  $x^*$ . As  $\gamma \uparrow n$ ,  $D^\gamma f(x)$  tends to  $f^{(n)}(x)$ , whereas as  $\gamma \downarrow n$  it tends to  $f^{(n)}(x) - f^{(n)}(0)$ . The representation **Equation 4** accounts for this switch automatically through  $\lceil \gamma(x) \rceil$ . If  $f^{(n)}(0) \neq 0$ , however,  $D^{\gamma(x)} f$  and hence the residual are discontinuous in  $x$ , which reduces the accuracy of a global polynomial collocation. All test problems of Section 5 in which an order crosses an integer satisfy  $f'(0) = 0$ , so that  $D^{\gamma(x)} f$  is continuous.

## 2.2 Shifted Chebyshev Polynomials

The shifted Chebyshev polynomials of the first kind are  $T_j^*(x) = T_j(2x - 1)$ ,  $x \in [0, 1]$ . Their monomial form is

$$T_j^*(x) = \sum_{m=0}^j a_{j,m} x^m, \quad a_{j,m} = (-1)^{j-m} \frac{2^{2m} j(j+m-1)!}{(j-m)!(2m)!}, \quad (5)$$

for  $j \geq 1$ , with  $a_{0,0} = 1$ . The case  $m = 0$  gives  $a_{j,0} = (-1)^j$ . For example,  $T_1^* = 2x - 1$  and  $T_2^* = 8x^2 - 8x + 1$ .

**Lemma 3** (Exact integral of the basis). For  $\gamma(x) \geq 0$ ,

$$I^{\gamma(x)} T_j^*(x) = \sum_{m=0}^j a_{j,m} \frac{\Gamma(m+1)}{\Gamma(m+1+\gamma(x))} x^{m+\gamma(x)}. \quad (6)$$

*Proof.* The operator is linear, and Lemma 1 applies to each monomial in **Equation 5**.  $\square$

For an integer order  $\gamma = l$ , **Equation 6** is the  $l$ -fold integral of  $T_j^*$  computed by direct monomial integration, since  $\Gamma(m+1)/\Gamma(m+1+l) = m!/(m+l)!$ .

### 3. COLLOCATION METHOD

The multiterm VOFDE considered is

$$D^{\alpha(x)} f(x) = h(x, f(x), D^{\beta_1(x)} f(x), \dots, D^{\beta_\nu(x)} f(x)), \quad x \in (0, 1], \quad (7)$$

with initial conditions

$$f^{(k)}(0) = f_k, \quad k = 0, \dots, l-1. \quad (8)$$

Here  $l$  satisfies  $l \geq [\alpha(x)]$  and  $l \geq [\beta_i(x)]$  for all  $x$  and  $i$ . The Caputo derivatives are taken in the sense of Definition 1.

The derivative  $u = D^l f$  is approximated by

$$u_N(x) = \sum_{j=0}^N c_j T_j^*(x) = C^T \Phi(x), \quad (9)$$

with  $C = [c_0, \dots, c_N]^T$  and  $\Phi = [T_0^*, \dots, T_N^*]^T$ . Integrating  $l$  times and using **Equation 8** gives

$$f_N(x) = \sum_{k=0}^{l-1} \frac{f_k}{k!} x^k + \sum_{j=0}^N c_j I^l T_j^*(x). \quad (10)$$

Every term of  $I^l T_j^*$  is a multiple of  $x^{m+l}$ , so the derivatives of orders  $0, \dots, l-1$  of the second sum vanish at  $x = 0$  and the initial conditions hold for every  $C$ . By Lemma 2, for any order  $\gamma \in \{\alpha, \beta_1, \dots, \beta_\nu\}$ ,

$$D^{\gamma(x)} f_N(x) = \psi_\gamma(x) + \sum_{j=0}^N c_j I^{l-\gamma(x)} T_j^*(x), \quad \psi_\gamma(x) = \sum_{k=\lceil \gamma(x) \rceil}^{l-1} \frac{f_k x^{k-\gamma(x)}}{\Gamma(k+1-\gamma(x))}, \quad (11)$$

where each  $I^{l-\gamma(x)} T_j^*$  is given in closed form by Lemma 3. The residual is

$$R_N(x; C) = D^{\alpha(x)} f_N(x) - h(x, f_N(x), D^{\beta_1(x)} f_N(x), \dots, D^{\beta_\nu(x)} f_N(x)). \quad (12)$$

It is required to vanish at the  $N+1$  Chebyshev-Gauss points of  $(0, 1)$ ,

$$x_i = \frac{1}{2} \left( 1 + \cos \frac{(2i+1)\pi}{2N+2} \right), \quad i = 0, \dots, N, \quad (13)$$

which gives the system

$$R_N(x_i; C) = 0, \quad i = 0, \dots, N. \quad (14)$$

This is a square system of  $N+1$  equations in  $N+1$  unknowns. Writing  $h = h(x, y_0, y_1, \dots, y_\nu)$  and denoting by  $\partial_k h$  its partial derivative with respect to  $y_k$ , the Jacobian is available analytically from **Equation 10** and **Equation 11**,

$$J_{ij} = I^{l-\alpha(x_i)} T_j^*(x_i) - \partial_0 h I^l T_j^*(x_i) - \sum_{k=1}^{\nu} \partial_k h I^{l-\beta_k(x_i)} T_j^*(x_i), \quad (15)$$

where the partial derivatives are evaluated at  $(x_i, f_N(x_i), D^{\beta_1(x_i)} f_N(x_i), \dots, D^{\beta_\nu(x_i)} f_N(x_i))$ . The system is solved by Newton's method, as summarized in **Algorithm 1**. When  $h$  is linear in its functional arguments,  $J$  does not depend on  $C$  and the system is solved directly as a linear system, so that Newton iteration is unnecessary.

**Remark 2** (Linear case). All benchmark problems in Section 5 have the form  $\sum_{s=0}^S q_s(x) D^{\gamma_s(x)} f(x) = g(x)$ , where  $\gamma_s \equiv 0$  represents the term in  $f$ . By **Equation 10** and **Equation 11**, the collocation system is  $MC = \mathbf{b}$  with

$$M_{ij} = \sum_{s=0}^S q_s(x_i) I^{l-\gamma_s(x_i)} T_j^*(x_i), \quad b_i = g(x_i) - \sum_{s=0}^S q_s(x_i) \psi_{\gamma_s}(x_i), \quad (16)$$

for  $i, j = 0, \dots, N$ , where  $\psi_0(x) = \sum_{k=0}^{l-1} f_k x^k / k!$ . The nonlinear branch of **Algorithm 1** is not exercised by the numerical tests reported here.

**Remark 3** (Conditioning of the monomial form). In the monomial form **Equation 5**, the sum of the absolute values of the coefficients of  $T_j^*$  equals  $T_j(3) \approx (3 + 2\sqrt{2})^j / 2$ . For example,  $T_{10}(3) \approx 2 \times 10^7$ . Evaluating **Equation 6** in double precision therefore loses accuracy by cancellation as  $N$  grows. The extent of this loss is quantified in Section 5.5, and the scheme as stated is intended for small and moderate  $N$ . For larger  $N$ , extended precision or a stable evaluation of the integrated basis is required.

**Algorithm 1** Chebyshev-Gauss collocation for multiterm VOFDEs

**Input:** orders  $\alpha(x), \beta_i(x)$ , function  $h$ , initial values  $f_0, \dots, f_{l-1}$ , degree  $N$ , tolerance  $\varepsilon$ .

1. Compute the nodes  $x_i$  from **Equation 13**.
2. Compute the coefficients  $a_{j,m}$  from **Equation 5** for  $0 \leq m \leq j \leq N$ .
3. Form  $I^l T_j^*$  and  $I^{l-\gamma(x)} T_j^*$  from **Equation 6** for  $\gamma \in \{\alpha, \beta_1, \dots, \beta_\nu\}$ , and  $\psi_\gamma$  from **Equation 11**.
4. Assemble  $R_N(x_i; C)$  from **Equation 12** and  $J$  from **Equation 15**.
5. Starting from  $C^{(0)} = 0$ , iterate  $C^{(r+1)} = C^{(r)} - J^{-1} \mathbf{R}(C^{(r)})$ , where  $\mathbf{R} = [R_N(x_0; C), \dots, R_N(x_N; C)]^T$ , until  $\|C^{(r+1)} - C^{(r)}\|_\infty < \varepsilon$ .
6. Return  $f_N$  from **Equation 10**.

## 4. ANALYSIS

### 4.1 Boundedness of the Variable-Order Integral

**Lemma 4.** Let  $\gamma : [0, 1] \rightarrow [\gamma_{\min}, \infty)$  be measurable with  $\gamma_{\min} > 0$ , and let  $M_\Gamma = 1/\min_{z>0} \Gamma(z) \approx 1.1292$ . Then  $I^{\gamma(x)}$  is bounded on  $L^2(0, 1)$  and

$$\|I^{\gamma(x)} v\|_{L^2(0,1)} \leq \frac{M_\Gamma}{\sqrt{\gamma_{\min}}} \|v\|_{L^2(0,1)}. \quad (17)$$

*Proof.* Write  $I^{\gamma(x)} v(x) = \int_0^x K(x, t) v(t) dt$  with  $K(x, t) = (x - t)^{\gamma(x)-1} / \Gamma(\gamma(x))$ . The gamma function attains its minimum on  $(0, \infty)$  at the positive zero  $z_0 = 1.4616 \dots$  of the digamma function, and  $\Gamma(z_0) \approx 0.8856$ . Hence both  $1/\Gamma(z)$  and  $1/\Gamma(z+1)$  do not exceed  $M_\Gamma$  for  $z > 0$ . For every  $x$ ,

$$\int_0^x |K(x, t)| dt = \frac{x^{\gamma(x)}}{\Gamma(\gamma(x) + 1)} \leq M_\Gamma.$$

For every  $t$ , since  $0 < x - t \leq 1$  and  $\gamma(x) \geq \gamma_{\min}$ , one has  $(x - t)^{\gamma(x)-1} \leq (x - t)^{\gamma_{\min}-1}$ , so

$$\int_t^1 |K(x, t)| dx \leq M_\Gamma \int_0^1 u^{\gamma_{\min}-1} du = \frac{M_\Gamma}{\gamma_{\min}}.$$

The Schur test gives the bound  $\sqrt{M_\Gamma \cdot M_\Gamma / \gamma_{\min}}$ . □

### 4.2 Exactness for Polynomial Solutions

**Proposition 1.** Let the exact solution  $f$  of **Equation 7** and **Equation 8** be a polynomial of degree at most  $N + l$ . Then  $f^{(l)} = \sum_{j=0}^N c_j^* T_j^*$  for a unique  $C^*$ ,  $f_N = f$  for this vector, and  $R_N(x; C^*) = 0$  for all  $x \in (0, 1]$ . In particular,  $C^*$  solves the collocation system **Equation 14**.

*Proof.* The polynomial  $f^{(l)}$  has degree at most  $N$  and lies in the span of  $T_0^*, \dots, T_N^*$ . **Equation 10** then reproduces  $f$  by Taylor's formula, and **Equation 11** reproduces every  $D^{\gamma(x)} f$  by Lemma 2. The residual **Equation 12** is therefore the difference of the two sides of **Equation 7** for the exact solution, which is zero. □

Whether the computed solution coincides with  $C^*$  depends on the uniqueness of the solution of **Equation 14**. For linear problems, uniqueness holds whenever the collocation matrix is nonsingular.

### 4.3 Error Estimate

Let  $\omega(x) = (x - x^2)^{-1/2}$  and let  $\|\cdot\|_\omega$  denote the norm of  $L_\omega^2(0, 1)$ . Since  $x - x^2 \leq 1/4$  on  $(0, 1)$ ,  $\omega \geq 2$  there, and therefore  $\|v\|_{L^2} \leq \|v\|_\omega / \sqrt{2}$ . The substitution  $x = (1 + t)/2$  maps the weighted integral to the Chebyshev-weighted integral on

$(-1, 1)$ , so the Gauss-Chebyshev quadrature with nodes **Equation 13** and equal weights  $\pi/(N+1)$  is exact for polynomials of degree at most  $2N+1$ . Hence, for  $v \in \mathbb{P}_N$ ,

$$\|v\|_{\omega}^2 = \frac{\pi}{N+1} \sum_{i=0}^N v(x_i)^2. \quad (18)$$

Let  $H_{\omega}^s(0, 1)$  be the weighted Sobolev space of Chebyshev spectral theory [20], transferred to  $(0, 1)$  by the affine map  $x \mapsto 2x - 1$  (the resulting scaling factors are absorbed in the constants), and let  $\mathcal{I}_N$  denote interpolation at the nodes **Equation 13**. For  $u \in H_{\omega}^s$  with  $s \geq 1$ , the interpolation estimates of [20] give

$$\|u - \mathcal{I}_N u\|_{\omega} \leq C_1 N^{-s} \|u\|_{H_{\omega}^s}, \quad \|u - \mathcal{I}_N u\|_{\infty} \leq C_2 N^{1/2-s} \|u\|_{H_{\omega}^s}. \quad (19)$$

The standing assumptions are the following. The orders  $\alpha, \beta_i$  are measurable with values in  $(0, l]$ , the function  $h$  is continuous in all arguments, and the exact solution satisfies  $f \in C^l[0, 1]$  with  $u = f^{(l)} \in H_{\omega}^s(0, 1)$ ,  $s \geq 1$ , so that  $u$  is continuous. In terms of  $u$ , the equation is  $\Phi(u) = 0$ , where, for continuous  $v$ ,

$$\begin{aligned} \Phi(v)(x) = & \psi_{\alpha}(x) + I^{l-\alpha(x)} v(x) \\ & - h\left(x, \psi_0(x) + I^l v(x), \psi_{\beta_1}(x) + I^{l-\beta_1(x)} v(x), \dots, \psi_{\beta_{\nu}}(x) + I^{l-\beta_{\nu}(x)} v(x)\right), \end{aligned} \quad (20)$$

and  $\psi_0(x) = \sum_{k < l} f_k x^k / k!$  is  $\psi_{\gamma}$  with  $\gamma = 0$ . The collocation solution satisfies  $\Phi(u_N)(x_i) = 0$ .

**Lemma 5.** Let  $h$  satisfy  $|h(x, y) - h(x, y')| \leq L \sum_{i=0}^{\nu} |y_i - y'_i|$ . Then for continuous  $v, w$ ,

$$\|\Phi(v) - \Phi(w)\|_{\infty} \leq K \|v - w\|_{\infty}, \quad K = M_{\Gamma} + L \left( \frac{1}{l!} + \nu M_{\Gamma} \right). \quad (21)$$

*Proof.* For  $e = v - w$  and any order  $\gamma$  with  $l - \gamma(x) \geq 0$ ,  $|I^{l-\gamma(x)} e(x)| \leq x^{l-\gamma(x)} \|e\|_{\infty} / \Gamma(l - \gamma(x) + 1) \leq M_{\Gamma} \|e\|_{\infty}$ , and  $|I^l e(x)| \leq \|e\|_{\infty} / l!$ . The terms  $\psi_{\gamma}$  and  $\psi_0$  cancel in the difference. Collecting the contributions of the  $\alpha$ -term and of the  $\nu + 1$  arguments of  $h$  gives  $K$ .  $\square$

**Theorem 1** (Conditional error estimate). Let  $u = f^{(l)} \in H_{\omega}^s(0, 1)$  with  $s \geq 1$ , let  $h$  satisfy the Lipschitz condition of Lemma 5, and let  $u_N \in \mathbb{P}_N$  solve **Equation 14**. Assume the following discrete stability hypothesis: there exist  $C_s > 0$  and  $N_0$  such that for  $N \geq N_0$  and all  $v, w \in \mathbb{P}_N$ ,

$$\|v - w\|_{\omega} \leq C_s \left( \frac{\pi}{N+1} \sum_{i=0}^N |\Phi(v)(x_i) - \Phi(w)(x_i)|^2 \right)^{1/2}. \quad (22)$$

Then, with  $\Lambda = (C_1 + C_s \sqrt{\pi} K C_2) / (\sqrt{2} l!)$ , which does not depend on  $N$ ,

$$\|f - f_N\|_{L^2(0,1)} \leq \frac{1}{l!} \|u - u_N\|_{L^2(0,1)} \leq \Lambda N^{1/2-s} \|u\|_{H_{\omega}^s}. \quad (23)$$

*Proof.* Since  $f - f_N = I^l(u - u_N)$  and the kernel  $(x - t)^{l-1} / (l-1)!$  has Schur bounds  $1/l!$  in both variables,  $\|f - f_N\|_{L^2} \leq \|u - u_N\|_{L^2} / l!$ . Because  $\|\cdot\|_{L^2} \leq \|\cdot\|_{\omega} / \sqrt{2}$ , it remains to bound  $\|u - u_N\|_{\omega}$ . Put  $\tilde{u} = \mathcal{I}_N u$ . Applying **Equation 22** with  $v = \tilde{u}$  and  $w = u_N$ , and using  $\Phi(u_N)(x_i) = 0 = \Phi(u)(x_i)$ ,

$$\begin{aligned} \|\tilde{u} - u_N\|_{\omega} & \leq C_s \left( \frac{\pi}{N+1} \sum_{i=0}^N |\Phi(\tilde{u})(x_i) - \Phi(u)(x_i)|^2 \right)^{1/2} \\ & \leq C_s \sqrt{\pi} K \|\tilde{u} - u\|_{\infty}, \end{aligned}$$

by Lemma 5. The triangle inequality and **Equation 19** give

$$\|u - u_N\|_{\omega} \leq C_1 N^{-s} \|u\|_{H_{\omega}^s} + C_s \sqrt{\pi} K C_2 N^{1/2-s} \|u\|_{H_{\omega}^s}.$$

Since  $N^{-s} \leq N^{1/2-s}$  for  $N \geq 1$ , the claim follows. The constant  $\Lambda$  depends on  $l, \nu, L, s, C_1, C_2$ , and  $C_s$ , and not on  $N$  or  $u$ .  $\square$

**Remark 4.** Hypothesis **Equation 22** is a property of the discrete problem and is not proved here, so Theorem 1 is conditional. For a linear problem it is equivalent to a uniform bound on the inverse of the collocation matrix in the scaled norm **Equation 18**, which can be monitored through the smallest singular value of that matrix. The same argument, applied to two data sets  $(h, f_k)$  and  $(\tilde{h}, \tilde{f}_k)$ , bounds  $\|u_N - \tilde{u}_N\|_{\omega}$  by  $C_s \sqrt{\pi}$  times a constant multiple of  $\sum_k |f_k - \tilde{f}_k| + \sup |h - \tilde{h}|$ , since  $\psi_{\gamma}$  is bounded by  $M_{\Gamma} \sum_k |f_k|$  on  $[0, 1]$ . This is the discrete counterpart of continuous dependence on the data. The rate  $N^{1/2-s}$  is not claimed to be sharp, and the numerical evidence of Section 5.5 indicates that it is pessimistic for the tested problems.

## 5. NUMERICAL RESULTS

### 5.1 Setup

All computations for the six benchmark problems were carried out in IEEE double precision arithmetic. The maximum absolute error  $E_{\max} = \max_x |f(x) - f_N(x)|$  was evaluated on a fine uniform grid of  $[0, 1]$ . Because every test problem is linear in  $C$ , the collocation system is solved directly by LU factorization, and no stopping tolerance is involved. In the six benchmark problems the exact solution is a polynomial, and the source term  $g$  is obtained from the identity

$$D^{\gamma(x)} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\gamma(x))} x^{p-\gamma(x)}, \quad p \in \mathbb{N}, \quad 0 < \gamma(x) \leq p, \quad (24)$$

which follows from Lemma 2, together with  $D^\gamma c = 0$  for constants  $c$  and  $\gamma > 0$ . The additional tests of Section 5.5 were computed in Python 3 with NumPy, SciPy, and mpmath, and the six benchmark problems were reproduced with the same code. The code is available from the corresponding author on reasonable request.

### 5.2 Example 1: Four-Term Equation

The equation, benchmarked in [12, 13, 14, 17], is

$$D^{\alpha(x)} f + x^{1/2} D^{\beta_1(x)} f + x^{1/3} D^{\beta_2(x)} f + x^{1/4} D^{\beta_3(x)} f + x^{1/5} f = g(x), \quad (25)$$

with  $f(0) = 2$  and  $f'(0) = 0$ , so that  $l = 2$ . The exact solution is  $f(x) = 2 - x^2/2$ . By **Equation 24**,

$$g(x) = -\frac{x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} - \frac{x^{5/2-\beta_1(x)}}{\Gamma(3-\beta_1(x))} - \frac{x^{7/3-\beta_2(x)}}{\Gamma(3-\beta_2(x))} - \frac{x^{9/4-\beta_3(x)}}{\Gamma(3-\beta_3(x))} + x^{1/5} \left(2 - \frac{x^2}{2}\right).$$

Problem (a) uses the variable orders  $\alpha(x) = 2x$ ,  $\beta_1(x) = x/3$ ,  $\beta_2(x) = x/4$ , and  $\beta_3(x) = x/5$ . Problem (b) uses the constant orders  $\alpha = 2$ ,  $\beta_1 = 1.234$ ,  $\beta_2 = 1$ , and  $\beta_3 = 0.333$ . Problem (a) is the benchmark of [12], and problem (b) coincides with Example 5 of [21] for its first parameter set.

With  $N = 2$ , Proposition 1 applies, since  $f'' = -1$  lies in the span of  $T_0^*$ , and the exact coefficient vector is  $C^* = (-1, 0, 0)^T$ . The order  $\alpha(x) = 2x$  crosses the value one at  $x = 1/2$ , and  $f'(0) = 0$ , so Remark 1 guarantees that  $D^{\alpha(x)} f$  remains continuous.

**Figure 1** shows the results for  $N = 2$ . In the upper panels the computed and exact solutions cannot be distinguished, and  $f$  decreases from 2 at  $x = 0$  to 1.5 at  $x = 1$ . The lower panels show absolute errors on a logarithmic axis with a floor of  $10^{-17}$ . For problem (a), the maximum error is  $1.11 \times 10^{-15}$ , which is five units of the double precision spacing  $2.2 \times 10^{-16}$  that applies to values in  $[1, 2]$ , and the error vanishes, within the plotting floor, for  $x > 0.9$ . For problem (b), the error is below the plotting floor for  $x < 0.43$  and grows toward  $x = 1$  with a peak near  $x \approx 0.95$  that reaches the  $10^{-14}$  decade; the maximum value is  $4.44 \times 10^{-14}$ . Since the discretization is exact for this problem, this growth is attributed to rounding in the evaluation of **Equation 6**, of the gamma function, and of the source term. This attribution is supported by a second implementation, which returned a maximum error of  $2.2 \times 10^{-16}$  for problem (b).

For problem (a), the cited works report errors at their own collocation points: 0 for  $n = 3$  and  $2.2204 \times 10^{-16}$  for  $n = 4$  in Table 1 of [12]; exact recovery for  $n = 2$  and, for  $l = 1$  and  $n = 3$  to 5, values between  $2.4 \times 10^{-18}$  and  $3.5 \times 10^{-17}$  in Tables 1 and 2 of [14]; and a maximum error of  $4 \times 10^{-15}$  for  $N = 2$ ,  $M = 3$  in [17]. **Table 1** compares problem (b). The entries for [15], [12], and [14] are quoted from Table 3 of [14], where  $n$  is the size parameter of that table, and the entry for [12] also appears in Table 2 of that work, where  $n = 3$  denotes degree three and thus four basis functions. The entry for [21] is the maximal error for step size 0.1 listed in the original report. The entry for [17] is the exact recovery reported there. No entry was recomputed.

The entries are not strictly comparable. The errors in [12, 14, 17] are measured at collocation points, whereas the present value is a maximum over a fine grid. The scheme of [21] is a first-order step-size method, and the value  $2.3659 \times 10^{-2}$  quoted for it in [14] for  $n = 10$  differs from the value in its original report, which is used in **Table 1**. In addition, the cited operational matrix works state the Caputo definition for orders in  $(0, 1]$  and apply the monomial rule for all  $m \geq 1$  [12, 14], so their treatment of orders above one is not identical to Definition 1. The two largest entries, from [21] and [15], are far above rounding level, whereas the remaining entries, including the present one, are at rounding level. The present value of  $4.44 \times 10^{-14}$  is larger than the value  $4.4409 \times 10^{-16}$  reported in [12] and than the exact recovery reported in [14] and [17]. Because the exact solution lies in the span of the present basis, the discretization error is zero, and the difference is attributed to rounding, as discussed for **Figure 1**. **Table 1** therefore does not indicate a difference in accuracy between the schemes. A basis of one function,  $N = 0$ , already contains the exact solution, so the basis size used here is not minimal.

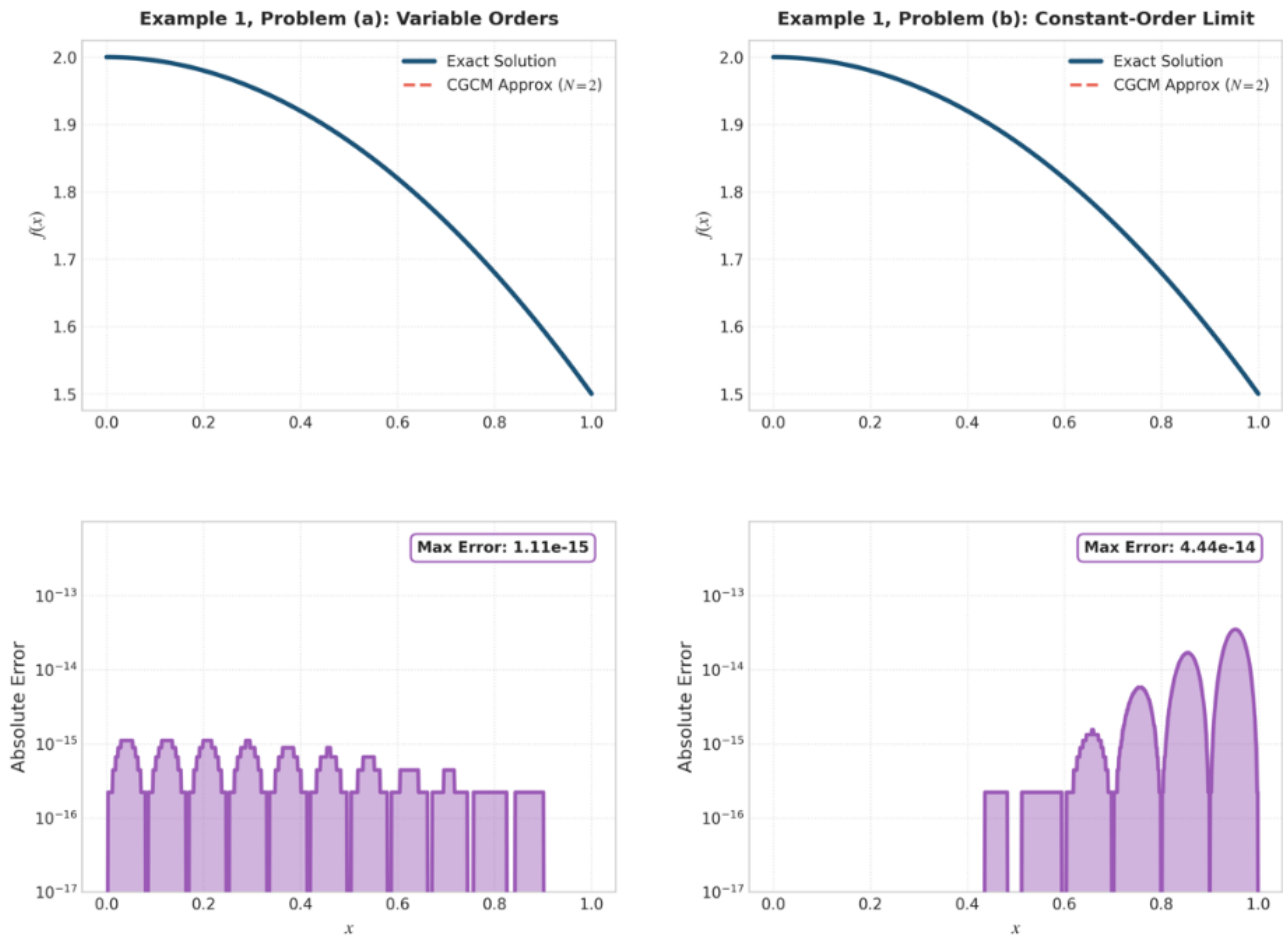


Figure 1: Exact and CGCM solutions with absolute errors for Example 1

Table 1: Maximum absolute error for Example 1, problem (b)

| Method                    | Size parameter               | Error sampling     | $E_{\max}$               |
|---------------------------|------------------------------|--------------------|--------------------------|
| El-Mesiry et al. [21]     | step size 0.1                | step grid          | $4.0960 \times 10^{-2}$  |
| Maleknejad et al. [15]    | $n = 5$                      | as quoted          | $6.8838 \times 10^{-5}$  |
| Liu et al. [12]           | $n = 3$ (4 functions)        | collocation points | $4.4409 \times 10^{-16}$ |
| El-Sayed and Agarwal [14] | $n = 2$ (3 functions)        | collocation points | exact                    |
| Postavaru and Toma [17]   | $N = 1, M = 1$ (2 functions) | collocation points | exact                    |
| Present method            | $N = 2$ (3 functions)        | fine grid          | $4.44 \times 10^{-14}$   |

### 5.3 Example 2: Three-Term Equation

The equation is

$$D^{\alpha(x)} f + q D^{\beta(x)} f + r f = g(x), \quad (26)$$

with  $f(0) = f_0$  and  $f'(0) = f_1$ , so that  $l = 2$  in all four problems. The exact solution is  $f(x) = x^2$  in problems (a), (c), and (d). The four problems are defined as follows.

**Problem (a):**  $\alpha(x) = 2x$ ,  $\beta(x) = (1+x)/2$ ,  $q = 2$ ,  $r = 4$ ,  $f_0 = f_1 = 0$ , and by **Equation 24**

$$g(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + \frac{4x^{2-\beta(x)}}{\Gamma(3-\beta(x))} + 4x^2.$$

**Problem (b):**  $\alpha(x) = (x + 2e^x)/7$ ,  $\beta(x) = 1$ ,  $q = -10$ ,  $r = 1$ ,  $f_0 = 5$ ,  $f_1 = 10$ , with exact solution  $f(x) = 5(1+x)^2$  and

$$g(x) = \frac{10x^{1-\alpha(x)}}{\Gamma(2-\alpha(x))} + \frac{10x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 5x^2 - 90x - 95.$$

The order  $\alpha(x)$  increases from  $2/7$  at  $x = 0$  to about  $0.92$  at  $x = 1$ , so it is non-polynomial and always below one, while two initial conditions are prescribed and  $l = 2$ . The equation was introduced in [11] with the condition  $f(0) = 5$  only, and the second condition is added in [14]. The exact solution satisfies both.

*Problem (c):*  $\alpha = 2, \beta = 1, q = 2, r = 4, f_0 = f_1 = 0$ , and  $g(x) = 2 + 4x + 4x^2$ .

*Problem (d), the Bagley-Torvik equation:*  $\alpha = 2, \beta = 3/2, q = r = 1, f_0 = f_1 = 0$ , and  $g(x) = x^2 + 2 + 4\pi^{-1/2}x^{1/2}$ .

With  $N = 2$  for problems (a), (c), and (d), and  $N = 3$  for problem (b), Proposition 1 applies, because  $f''$  is constant. The exact coefficient vectors are  $(2, 0, 0)^T$  for problems (a), (c), and (d), and  $(10, 0, 0, 0)^T$  for problem (b). The results are shown in **Figure 2**.

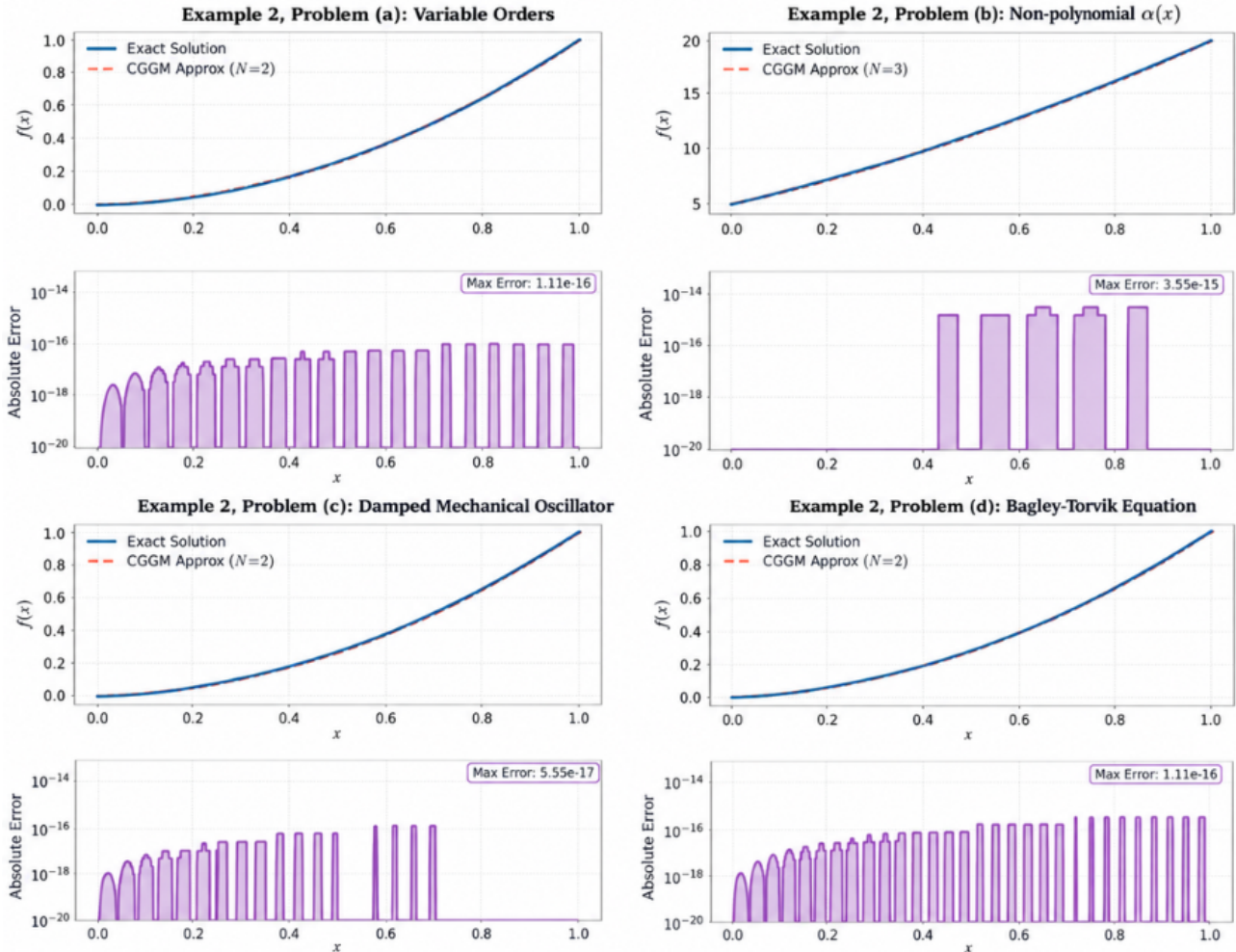


**Figure 2: Exact and CGCM solutions with absolute errors for Example 2**

The maximum errors in **Figure 2** are  $1.11 \times 10^{-16}$ ,  $3.55 \times 10^{-15}$ ,  $5.55 \times 10^{-17}$ , and  $1.11 \times 10^{-16}$  for problems (a) to (d). For problems (a), (c), and (d),  $f$  lies in  $[0, 1]$ , where the double precision spacing is at most  $1.1 \times 10^{-16}$ , so these errors amount to at most one unit in the last place of the exact solution. For problem (c) the error vanishes within the plotting floor for  $x > 0.7$ . For problem (b),  $f$  increases from 5 to 20, and the error of  $3.55 \times 10^{-15}$  equals the spacing of double precision numbers in  $[16, 32)$ . The error appears only for  $0.43 \leq x \leq 0.86$  and is not visible at the plotting floor elsewhere. The computed solution is therefore correct to the resolution of the number format.

For problem (b), **Table 2** compares the maximum error with published values. The entries for [11] and [14] are the maxima of the pointwise errors reported at  $x = 0.2, \dots, 1.0$  in Table 1 of [11] and Table 5 of [14], and were not recomputed. The entry for [17] is the maximum error stated in that work for  $N = 2, M = 1$ .

The present value is lower than that of the Legendre wavelet method, which uses 16 functions, by a factor of about  $6 \times 10^5$ , and lower than that of the shifted Legendre method, which uses 3 functions, by a factor of about  $7 \times 10^2$ . These ratios do not indicate a more accurate method. The present basis contains the exact solution, whereas the cited errors are reported for the bases and sizes stated in **Table 2**, and the sources of those errors were not analyzed here. The difference from [17] is a factor of two at the  $10^{-15}$  level and has no practical significance, because both results are at the rounding level. For the Bagley-Torvik problem, [14] reports exact recovery with three basis functions and quotes errors of order  $10^{-10}$  for the

**Table 2: Maximum absolute error for Example 2, problem (b)**

| Method                    | Size parameter                | Error sampling | $E_{\max}$             |
|---------------------------|-------------------------------|----------------|------------------------|
| Chen et al. [11]          | $k = 2, M = 4$ (16 functions) | five points    | $2.02 \times 10^{-9}$  |
| El-Sayed and Agarwal [14] | $n = 2$ (3 functions)         | five points    | $2.43 \times 10^{-12}$ |
| Postavaru and Toma [17]   | $N = 2, M = 1$ (4 functions)  | as reported    | $7 \times 10^{-15}$    |
| Present method            | $N = 3$ (4 functions)         | fine grid      | $3.55 \times 10^{-15}$ |

scheme of [22].

## 5.4 Summary of the Polynomial Benchmarks

**Table 3** shows the maximum errors for all six problems.

**Table 3: Maximum absolute error of the present method for the six benchmark problems**

| Problem      | $N$ | Basis size | $E_{\max}$             |
|--------------|-----|------------|------------------------|
| Example 1(a) | 2   | 3          | $1.11 \times 10^{-15}$ |
| Example 1(b) | 2   | 3          | $4.44 \times 10^{-14}$ |
| Example 2(a) | 2   | 3          | $1.11 \times 10^{-16}$ |
| Example 2(b) | 3   | 4          | $3.55 \times 10^{-15}$ |
| Example 2(c) | 2   | 3          | $5.55 \times 10^{-17}$ |
| Example 2(d) | 2   | 3          | $1.11 \times 10^{-16}$ |

A second double precision implementation based on **Equation 16**, with the source terms given above, reproduced the coefficient vectors  $C^*$  with maximum errors not exceeding  $7.1 \times 10^{-15}$ . The condition number of the collocation matrix did not exceed  $4.3 \times 10^2$  for any problem. The exact error values in **Table 3** therefore depend on the evaluation order, and only their rounding-level magnitude is meaningful.

Since all six exact solutions are polynomials, Proposition 1 predicts rounding-level errors. The experiments verify the implementation of the exact integration for variable, non-polynomial, and constant orders, including orders that cross an integer value, as  $\alpha(x) = 2x$  does at  $x = 1/2$ . These six problems do not test convergence in  $N$  and do not show that the present basis is more accurate than the compared bases for solutions outside the span of the basis. This limitation is addressed in the next subsection.

## 5.5 Non-Polynomial Solutions, Convergence, and Conditioning

Two tests are defined with the operator of Example 2(a), namely  $D^{\alpha(x)} f + 2D^{\beta(x)} f + 4f = g(x)$  with  $\alpha(x) = 2x$ ,  $\beta(x) = (1+x)/2$ ,  $l = 2$ , and  $f(0) = f'(0) = 0$ . The exact solutions are  $f(x) = x^2 e^x$ , which is entire, and  $f(x) = x^{5/2}$ , for which  $f'' = \frac{15}{4}x^{1/2}$  is singular at  $x = 0$ . Termwise application of **Equation 24** to  $x^2 e^x = \sum_{k \geq 0} x^{k+2}/k!$  gives

$$D^{\gamma(x)}(x^2 e^x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+3)}{k! \Gamma(k+3-\gamma(x))} x^{k+2-\gamma(x)},$$

and  $D^{\gamma(x)} x^{5/2} = \Gamma(7/2) x^{5/2-\gamma(x)} / \Gamma(7/2-\gamma(x))$ . Both formulas hold for  $0 < \gamma \leq 2$ , and both solutions satisfy  $f'(0) = 0$ . The source terms were evaluated with 60 significant digits. The errors were measured at 200 uniform points of  $(0, 1]$ , in double precision and in 60-digit arithmetic. The latter isolates the truncation error from rounding in the monomial representation.

**Table 4** lists the results for  $f = x^2 e^x$ . The extended precision error decreases by about three orders of magnitude for every increase of  $N$  by two at small  $N$  and by about four orders at larger  $N$ , from  $2.54 \times 10^{-2}$  at  $N = 2$  to  $1.43 \times 10^{-32}$  at  $N = 20$ , which indicates exponential convergence. The double precision error follows the extended precision error down to  $N = 10$  and then remains between  $1.3 \times 10^{-15}$  and  $2.7 \times 10^{-15}$  up to  $N = 24$ . It rises to  $1.42 \times 10^{-12}$  at  $N = 28$ . The condition

number  $\kappa(M)$  of the collocation matrix, computed from the matrix assembled in double precision, grows moderately, from  $1.3 \times 10^2$  at  $N = 2$  to  $7.7 \times 10^4$  at  $N = 20$ , whereas  $T_N(3)$  of Remark 3 grows geometrically and reaches  $10^{15}$  at  $N = 20$ . The jump of the computed  $\kappa(M)$  for  $N \geq 24$  reflects rounding in the assembled matrix and not a property of the continuous problem.

**Table 4: Errors, conditioning, and cancellation measure for  $f(x) = x^2 e^x$**

| $N$ | $E_{\max}$ (double)    | $E_{\max}$ (60 digits) | $\kappa(M)$          | $T_N(3)$             |
|-----|------------------------|------------------------|----------------------|----------------------|
| 2   | $2.54 \times 10^{-2}$  | $2.54 \times 10^{-2}$  | $1.3 \times 10^2$    | $1.7 \times 10^1$    |
| 4   | $5.62 \times 10^{-5}$  | $5.62 \times 10^{-5}$  | $7.6 \times 10^2$    | $5.8 \times 10^2$    |
| 6   | $6.33 \times 10^{-8}$  | $6.33 \times 10^{-8}$  | $2.3 \times 10^3$    | $2.0 \times 10^4$    |
| 8   | $4.55 \times 10^{-11}$ | $4.55 \times 10^{-11}$ | $5.3 \times 10^3$    | $6.7 \times 10^5$    |
| 10  | $2.42 \times 10^{-14}$ | $2.41 \times 10^{-14}$ | $1.0 \times 10^4$    | $2.3 \times 10^7$    |
| 12  | $2.66 \times 10^{-15}$ | $9.21 \times 10^{-18}$ | $1.7 \times 10^4$    | $7.7 \times 10^8$    |
| 16  | $1.33 \times 10^{-15}$ | $5.71 \times 10^{-25}$ | $3.9 \times 10^4$    | $8.9 \times 10^{11}$ |
| 20  | $1.33 \times 10^{-15}$ | $1.43 \times 10^{-32}$ | $7.7 \times 10^4$    | $1.0 \times 10^{15}$ |
| 24  | $1.33 \times 10^{-15}$ | $1.78 \times 10^{-40}$ | $1.5 \times 10^8$    | $1.2 \times 10^{18}$ |
| 28  | $1.42 \times 10^{-12}$ | $1.19 \times 10^{-48}$ | $2.6 \times 10^{14}$ | $1.4 \times 10^{21}$ |

**Table 5** lists the results for  $f = x^{5/2}$ . The function  $u = f''$  has an endpoint singularity of type  $x^{1/2}$ , whose Chebyshev coefficients decay like  $n^{-2}$ , so that  $u \in H_\omega^s$  for every  $s < 3/2$ . Theorem 1 would then give a rate arbitrarily close to  $N^{-1}$  under hypothesis **Equation 22**. The observed order  $p_N = \ln(E_{N_{\text{prev}}}/E_N)/\ln(N/N_{\text{prev}})$  increases from 2.68 to 3.50 over  $2 \leq N \leq 48$ , so the estimate is consistent with the data but pessimistic. The tested range is too short to decide whether  $p_N$  settles at a limiting value. The double precision error coincides with the extended precision error to three digits up to  $N = 18$ , deviates at  $N = 20$  ( $1.68 \times 10^{-5}$  against  $1.27 \times 10^{-5}$ ), and is lost at  $N = 24$  ( $8.8 \times 10^{-3}$ ). At  $N = 32$  the double precision solution is no longer meaningful. The two tests indicate that double precision is reliable up to about  $N = 18$  for the problem of limited regularity and up to about  $N = 24$  for the smooth problem. These thresholds are specific to the tested problems and should not be extrapolated.

**Table 5: Errors and observed order for  $f(x) = x^{5/2}$**

| $N$ | $E_{\max}$ (60 digits) | Observed order $p_N$ | $E_{\max}$ (double)   |
|-----|------------------------|----------------------|-----------------------|
| 2   | $1.85 \times 10^{-2}$  | n/a                  | $1.85 \times 10^{-2}$ |
| 4   | $2.90 \times 10^{-3}$  | 2.68                 | $2.90 \times 10^{-3}$ |
| 8   | $2.96 \times 10^{-4}$  | 3.29                 | $2.96 \times 10^{-4}$ |
| 16  | $2.74 \times 10^{-5}$  | 3.43                 | $2.74 \times 10^{-5}$ |
| 32  | $2.49 \times 10^{-6}$  | 3.46                 | $4.59 \times 10^2$    |
| 48  | $6.02 \times 10^{-7}$  | 3.50                 | $4.44 \times 10^6$    |

**Figure 3** summarizes both tests. Panel (a) shows the maximum errors against  $N$  in both arithmetics, and panel (b) shows  $\kappa(M)$  and  $T_N(3)$  for the smooth test. The exponential decay of the smooth case and the algebraic decay of the singular case are visible in panel (a), as is the departure of the double precision curves from the extended precision curves.

Per Newton iteration, forming the Jacobian and solving the dense  $(N + 1) \times (N + 1)$  system costs  $O(N^3)$  operations. No timing comparison is reported, because the cited works do not provide timings measured under the same conditions.

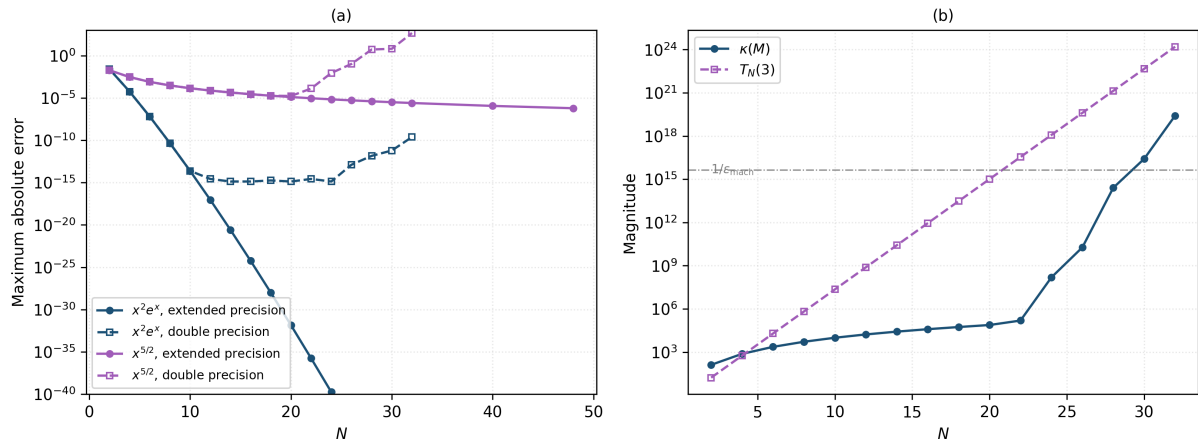


**Figure 3: Convergence and conditioning for non-polynomial solutions**

## 6. CONCLUSION

A Chebyshev-Gauss collocation method with exact variable-order fractional integration was developed for multiterm Caputo VOFDEs. The variable-order integral of the shifted Chebyshev basis is available in closed form, the initial conditions hold by construction, and the collocation system is square. The integral operator is bounded on  $L^2(0, 1)$  with the explicit constant  $M_\Gamma/\sqrt{\gamma_{\min}}$ , and polynomial solutions of degree up to  $N + l$  are recovered exactly. An  $N^{1/2-s}$  error estimate was derived under a discrete stability hypothesis that remains unproved, so the estimate is conditional.

Six linear benchmark problems with polynomial solutions, covering variable, constant, and non-polynomial orders, gave maximum errors from  $5.55 \times 10^{-17}$  to  $4.44 \times 10^{-14}$  with three or four basis functions. These values are at the rounding level and comparable with published operational matrix and hybrid block-pulse results, but polynomial benchmarks do not establish an accuracy advantage over those schemes. The largest error, for Example 1(b), exceeds the  $10^{-16}$  level reported elsewhere for that problem and is attributed to rounding. Two non-polynomial tests showed exponential convergence for an entire solution and an observed algebraic rate of about  $N^{-3.5}$  for a solution with an endpoint singularity in the second derivative, which is faster than the conditional estimate predicts. The monomial representation limits double precision arithmetic to roughly  $N = 24$  and  $N = 18$  in these two tests, respectively.

Two limitations remain. The stability hypothesis is unproved, and all tests were linear, so the nonlinear branch of the algorithm is not verified numerically. Future work should address a proof of the stability hypothesis for representative problem classes, a stable evaluation of the integrated basis for larger  $N$ , nonlinear and nonsmooth test problems, and the extension to variable-order fractional partial differential equations.

## AUTHOR CONTRIBUTION STATEMENT

All authors contributed equally to the study conception and design. Material preparation, data collection, and analysis were performed by the authors. The first draft of the manuscript was written by the authors, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

## ETHICS APPROVAL AND CONSENT TO PARTICIPATE

Ethics declaration: not applicable. This study did not involve human participants or animals.

## CONSENT FOR PUBLICATION

Not Applicable.

## DATA AVAILABILITY

All data generated or analysed during this study are included in this published article. The code is available from the corresponding author on reasonable request.

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## DISCLOSURE STATEMENT

The authors declare no conflict of interest.

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