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Classical and Bayesian Inference for Stress-Strength Reliability of Shifted Geometric Lifetimes under Unified Hybrid Censoring

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ABSTRACT

Stress-strength reliability inference is developed for independent shifted geometric strength and stress lifetimes observed under unified hybrid censoring schemes (UHCSs). The reliability $R = \mathbb{P}(X > Y)$ is appropriate for experiments in which lifetime is recorded as a number of operating cycles, inspections, or trials. Under the UHCS the likelihood reduces to a kernel in the number of observed failures and the total discrete exposure, which yields closed-form maximum likelihood estimators of the geometric parameters and, by invariance, of R . Wald and delta-method intervals are derived from the observed Fisher information, a conservative conditional beta interval is obtained from beta tails with a Bonferroni adjustment, and an exact procedure based on inversion of the UHCS sampling distribution is outlined. Bayesian inference under the Jeffreys prior of the shifted geometric model and under the uniform prior leads to beta posteriors for the parameters and to Monte Carlo evaluation of the posterior of R . A parametric bootstrap percentile interval is also considered. A pilot Monte Carlo study with $p_X = 0.05$, $p_Y = 0.10$, and sample sizes between 250 and 600 compares bias, mean squared error, coverage probability, and average interval length. The maximum likelihood and Jeffreys estimators are nearly indistinguishable, and the uniform prior adds a small bias. The conditional beta interval attains coverage of at least 0.995 with intervals about 1.6 times longer, whereas the delta-method, Jeffreys credible, and bootstrap intervals have coverage within Monte Carlo error of 0.95. Accelerated capacitor lifetimes converted to inspection cycles illustrate the methods. The estimate of R remains between 0.656 and 0.677 as the cycle width varies from 1 to 200 hours, while the per-cycle parameters change by a factor of about 180.

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1. INTRODUCTION

Stress-strength reliability measures the probability that the strength of a component exceeds the stress applied to it. For independent strength X and stress Y measured on a common scale, the measure is $R = \mathbb{P}(X > Y)$. The theory of the stress-strength model and its extensions is reviewed in [1]. Inference for R has been developed under many lifetime models and sampling schemes, including the geometric [2, 3], Kumaraswamy [4], generalized inverted exponential [5], Pareto

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[6, 7], Weibull [8], Poisson-exponential [9], and discrete alpha-power Weibull [10] distributions. Reliability measures of a bivariate lifetime model built from a Sarmanov copula have also been derived [11].

Complete lifetime data are often expensive or time-consuming to obtain, and censoring is therefore used to shorten life tests while retaining information on the lifetime distribution [12, 13]. Foundations of likelihood-based inference are given in [14]. Hybrid censoring combines failure-count and time-based termination rules. The unified hybrid censoring scheme (UHCS) terminates a test according to two prescribed failure counts and two prescribed times, and exact likelihood inference under this scheme was studied for the exponential distribution in [15].

When lifetime is recorded as a number of cycles, inspections, or attempts, the geometric distribution is a natural and tractable model. It is memoryless, has a constant discrete hazard, and leads to simple likelihoods under censoring [16, 17]. Stress-strength inference for geometric populations was studied in [2], extended to lower record values in [18], and treated by objective Bayesian methods based on Jeffreys, reference, and probability-matching priors in [3]. Record-value sampling, in which only values exceeding or falling below all earlier observations are retained, has been studied for the generalized Rayleigh distribution [19] and the power Lindley distribution [20]. Upper records were used for stress-strength inference in [4, 5, 6, 7], and Weibull record data with inter-record times were treated in [8]. Bayesian analysis of the Poisson-exponential distribution was developed in [21]. The discrete alpha-power Weibull distribution was introduced in [22], and its stress-strength reliability was studied under Type-II censored, upper record, and mixed upper-lower record data in [10].

Among the works cited above, none treats the shifted geometric stress-strength model under UHCS. This paper addresses that case, and its contributions are as follows:

1. The UHCS termination rule is stated precisely for discrete lifetimes, including a convention for ties.
2. Closed-form maximum likelihood estimators of the geometric parameters and of R are derived together with Wald, delta-method, and conservative conditional beta intervals.
3. Bayesian estimators under the Jeffreys and uniform priors are obtained, with beta posteriors for the parameters and Monte Carlo evaluation of the posterior of R .
4. A parametric bootstrap interval is described, and a pilot Monte Carlo study compares the estimators and intervals.
5. Accelerated capacitor lifetime data are converted to inspection-cycle counts, and the sensitivity of the results to the discretization width is examined.

The remainder of the paper is organized as follows. **Section 2** defines the model and the censoring scheme. **Section 3** derives the maximum likelihood estimators, and **Section 4** studies the reliability measure. **Section 5** presents interval estimators, **Section 6** the Bayesian analysis, and **Section 7** the bootstrap procedures. **Section 8** and **Section 9** describe and report the simulation study. **Section 10** analyzes the capacitor data, and **Section 11** and **Section 12** give the discussion and conclusions.

2. SHIFTED GEOMETRIC MODEL AND UNIFIED HYBRID CENSORING

Let X follow a shifted geometric distribution with parameter $p_X \in (0, 1)$, that is, a geometric distribution on the support $\{1, 2, \dots\}$ counting the number of cycles up to and including the first failure, with probability mass function

$$\mathbb{P}(X = x) = p_X(1 - p_X)^{x-1}, \quad x = 1, 2, \dots \quad (1)$$

Write $q_X = 1 - p_X$. Then

$$\mathbb{P}(X > t) = q_X^t, \quad t = 0, 1, 2, \dots, \quad \mathbb{E}(X) = \frac{1}{p_X}, \quad \text{Var}(X) = \frac{1 - p_X}{p_X^2}, \quad (2)$$

and the discrete hazard is constant, $\mathbb{P}(X = x \mid X \geq x) = p_X$. The variable Y is defined in the same way with parameter p_Y and $q_Y = 1 - p_Y$. The variables X and Y are independent.

Suppose n independent units from the X population are placed on test, and let $X_{1:n} \leq \dots \leq X_{n:n}$ denote the ordered lifetimes. Fix integers $1 \leq k < r \leq n$ and times $T_1 < T_2$. Under the UHCS, the test terminates at

$$\tau = \begin{cases} T_1, & X_{r:n} \leq T_1, \\ X_{r:n}, & X_{k:n} \leq T_1 < X_{r:n} \leq T_2, \\ T_2, & X_{k:n} \leq T_1 < T_2 < X_{r:n}, \\ X_{r:n}, & T_1 < X_{k:n} \leq X_{r:n} \leq T_2, \\ T_2, & T_1 < X_{k:n} \leq T_2 < X_{r:n}, \\ X_{k:n}, & T_2 < X_{k:n}. \end{cases} \quad (3)$$

For $k < r$ and $T_1 < T_2$, the six cases are mutually exclusive and exhaustive. Because lifetimes are integers, ties with the termination time must be resolved explicitly. A unit is treated as failed within the test if $X_i \leq \tau$, so that all units failing in the terminal cycle are observed. The number of observed failures is therefore $H = \#\{i : X_i \leq \tau\}$, which may exceed r or k when ties occur at $X_{r:n}$ or $X_{k:n}$. For example, if several units fail in the same cycle as $X_{r:n}$ and that cycle ends the test, all of these units are counted and $H > r$. In every case $\tau \geq X_{k:n}$, so that $H \geq k \geq 1$. The same rule, with its own constants (k, r, T_1, T_2) , is applied to the Y sample of size m and yields τ_Y and H_Y .

3. LIKELIHOOD AND MAXIMUM LIKELIHOOD ESTIMATION

An observed failure at cycle x_i contributes $p_X(1 - p_X)^{x_i-1}$ to the likelihood, and a unit still operating at the terminal time contributes $\mathbb{P}(X > \tau_X) = (1 - p_X)^{\tau_X}$. Let H_X denote the number of observed failures in the X sample, with failure cycles $x_1, \dots, x_{H_X}$. Define the total exposure, that is, the number of cycles survived before failure or termination by all units, as

$$A_X = \sum_{i=1}^{H_X} (x_i - 1) + (n - H_X) \tau_X. \quad (4)$$

Each failed unit contributes $x_i - 1$ completed cycles, and each unit still operating at termination contributes τ_X cycles.

The termination time τ_X and the failure count H_X are data dependent. Both are determined by the failure history observed up to τ_X , so that τ_X is a stopping time of the observed failure process. The probability of the observed data is therefore the product of the failure probabilities and the survival probabilities of the units still operating at τ_X , multiplied by a constant that does not involve p_X . The likelihood is thus proportional to

$$L_X(p_X) \propto p_X^{H_X} (1 - p_X)^{A_X}, \quad (5)$$

and in the same way $L_Y(p_Y) \propto p_Y^{H_Y} (1 - p_Y)^{A_Y}$ with failure cycles $y_1, \dots, y_{H_Y}$ and $A_Y = \sum_{j=1}^{H_Y} (y_j - 1) + (m - H_Y) \tau_Y$. Independence of the samples gives $L(p_X, p_Y) = L_X(p_X) L_Y(p_Y)$.

The log-likelihood for the strength parameter is $\ell_X(p_X) = H_X \log p_X + A_X \log(1 - p_X)$, with score

$$\frac{\partial \ell_X}{\partial p_X} = \frac{H_X}{p_X} - \frac{A_X}{1 - p_X}. \quad (6)$$

Setting the score to zero yields the maximum likelihood estimators

$$\hat{p}_X = \frac{H_X}{H_X + A_X}, \quad \hat{p}_Y = \frac{H_Y}{H_Y + A_Y}. \quad (7)$$

The condition $H_X \geq 1$ and $H_Y \geq 1$ holds automatically under the UHCS, since $H \geq k \geq 1$. The boundary value $\hat{p}_X = 1$ occurs only if $A_X = 0$, that is, if every unit fails in the first cycle, and is excluded from the parameter space. The second derivative is $-H_X/p_X^2 - A_X/(1 - p_X)^2 < 0$, so the stationary point is a maximum. The observed Fisher information evaluated at $\hat{p}_X$ is

$$I_X(\hat{p}_X) = \frac{H_X}{\hat{p}_X^2} + \frac{A_X}{(1 - \hat{p}_X)^2} = \frac{(H_X + A_X)^3}{H_X A_X}, \quad (8)$$

and the estimated variance is therefore

$$\widehat{\text{Var}}(\hat{p}_X) = \frac{H_X A_X}{(H_X + A_X)^3}. \quad (9)$$

The corresponding expressions for p_Y are obtained by replacing H_X and A_X with H_Y and A_Y .

4. STRESS-STRENGTH RELIABILITY

Under independence, the reliability is

$$R = \mathbb{P}(X > Y) = \sum_{y=1}^{\infty} \mathbb{P}(X > y) \mathbb{P}(Y = y) = p_Y q_X \sum_{y=1}^{\infty} (q_X q_Y)^{y-1}. \quad (10)$$

Summing the geometric series, which converges because $q_X q_Y < 1$, gives

$$R = \frac{(1 - p_X) p_Y}{p_X + p_Y - p_X p_Y}. \quad (11)$$

By the invariance property of maximum likelihood estimation, $\hat{R}_{ML}$ is obtained by substituting $\hat{p}_X$ and $\hat{p}_Y$ into **Equation 11**.

Let $D = p_X + p_Y - p_X p_Y$. The partial derivatives of R are

$$\frac{\partial R}{\partial p_X} = -\frac{p_Y}{D^2} < 0, \quad \frac{\partial R}{\partial p_Y} = \frac{p_X(1 - p_X)}{D^2} > 0. \quad (12)$$

A larger p_X shortens the expected strength lifetime and lowers R , whereas a larger p_Y shortens the expected stress lifetime and raises R . The normalized elasticities follow from **Equation 11** and **Equation 12** as

$$\frac{p_X}{R} \frac{\partial R}{\partial p_X} = -\frac{p_X}{(1 - p_X)D}, \quad \frac{p_Y}{R} \frac{\partial R}{\partial p_Y} = \frac{p_X}{D}. \quad (13)$$

At $p_X = 0.05$ and $p_Y = 0.10$, $D = 0.145$ and $R = 0.655172$, and the elasticities equal -0.363 and 0.345 . A relative increase of one percent in p_X therefore lowers R by about 0.36 percent, and a relative increase of one percent in p_Y raises R by about 0.34 percent. The level curves of R over the (p_X, p_Y) plane are shown in **Figure 1**.

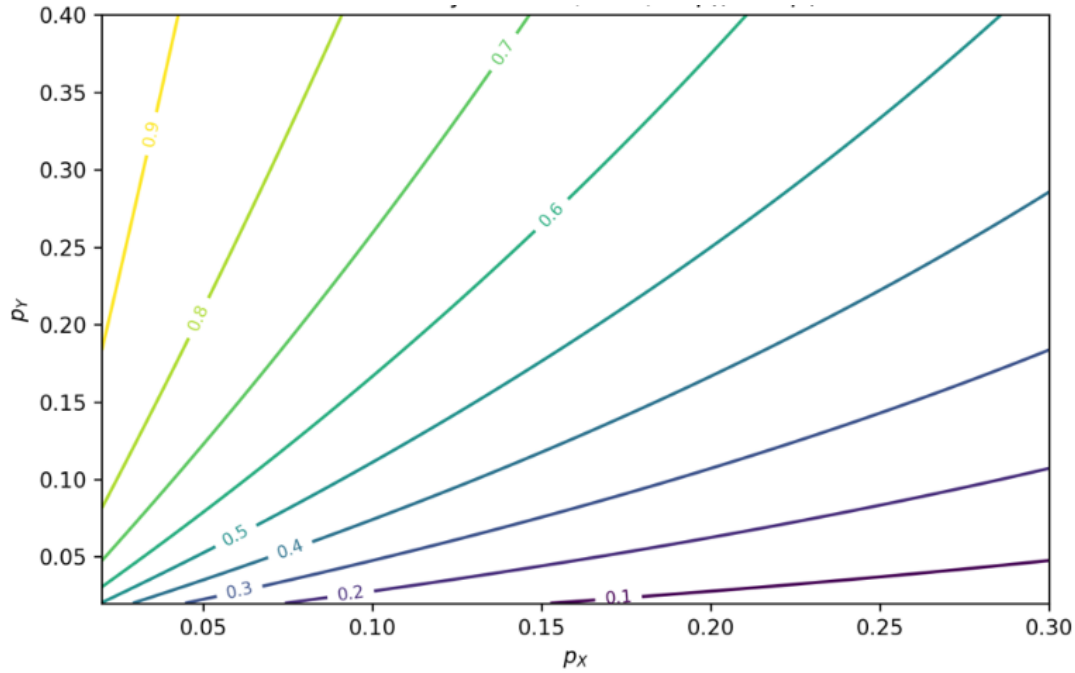


Figure 1: Contour Plot of $R = \mathbb{P}(X > Y)$ as a Function of p_X and p_Y

5. INTERVAL ESTIMATION

5.1 Wald and Delta-Method Intervals

A two-sided $100(1 - \alpha)\%$ Wald interval for p_X is

$$\hat{p}_X \pm z_{1-\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{p}_X)}, \quad (14)$$

with $\widehat{\text{Var}}(\hat{p}_X)$ from **Equation 9**, truncated to $[0, 1]$. The same construction applies to p_Y . For R , let $g_X = -\hat{p}_Y/\hat{D}^2$ and $g_Y = \hat{p}_X(1 - \hat{p}_X)/\hat{D}^2$ be the partial derivatives in **Equation 12** evaluated at $(\hat{p}_X, \hat{p}_Y)$, with $\hat{D} = \hat{p}_X + \hat{p}_Y - \hat{p}_X \hat{p}_Y$. Because the two samples are independent, the delta method gives

$$\widehat{\text{Var}}(\hat{R}_{ML}) = g_X^2 \widehat{\text{Var}}(\hat{p}_X) + g_Y^2 \widehat{\text{Var}}(\hat{p}_Y). \quad (15)$$

The delta-method interval, truncated to $[0, 1]$, is

$$\hat{R}_{ML} \pm z_{1-\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{R}_{ML})}. \quad (16)$$

5.2 Conservative Conditional Beta Interval

The kernel $p_X^{H_X}(1 - p_X)^{A_X}$ is that of H_X successes in $H_X + A_X$ Bernoulli trials. Beta-tail limits of the Clopper-Pearson form are therefore

$$L_X = B_{\alpha^*/2}^{-1}(H_X, A_X + 1), \quad U_X = B_{1-\alpha^*/2}^{-1}(H_X + 1, A_X), \quad (17)$$

where $B_u^{-1}(a, b)$ is the inverse of the beta cumulative distribution function with parameters a and b evaluated at u , and L_Y , U_Y are defined likewise, with the conventions $L = 0$ if $H = 0$ and $U = 1$ if $A = 0$. The nominal level $1 - \alpha^*$ is guaranteed only conditionally on $H_X + A_X$, which is treated as fixed, whereas under the adaptive UHCS rule this quantity is random. The interval is therefore called a conditional beta interval, and its coverage under the full stopping rule is examined by simulation rather than assumed. Since R decreases in p_X and increases in p_Y by **Equation 12**, component intervals at level $1 - \alpha/2$ each, that is $\alpha^* = \alpha/2$, give by the Bonferroni inequality the conservative interval

$$R_L = R(U_X, L_Y), \quad R_U = R(L_X, U_Y), \quad (18)$$

with nominal joint level at least $1 - \alpha$.

5.3 Exact Procedure by Test Inversion

An exact interval must account for the random stopping mechanism. For a candidate value p_0 , let s_t denote the number of units alive at the start of cycle t . Given s_t , the number of failures D_t in cycle t satisfies $D_t \mid s_t \sim \text{Binomial}(s_t, p_0)$. The distribution of the terminal state is obtained by dynamic programming as follows:

1. start with n surviving units at cycle 1;
2. enumerate the possible numbers of failures in the current cycle with the corresponding binomial probabilities;
3. update the cumulative failures, the survivors, and the accumulated exposure;
4. apply the UHCS termination rule in **Equation 3** with the tie convention;
5. compute $(H, A, \hat{p})$ and the probability of every terminal state; and
6. order the terminal states by $\hat{p}$ and compute the tail probabilities $\mathbb{P}_{p_0}(\hat{p} \geq \hat{p}_{\text{obs}})$ and $\mathbb{P}_{p_0}(\hat{p} \leq \hat{p}_{\text{obs}})$.

The exact confidence set for p is

$$\left\{ p_0 \in (0, 1) : \mathbb{P}_{p_0}(\hat{p} \geq \hat{p}_{\text{obs}}) > \frac{\alpha}{2} \text{ and } \mathbb{P}_{p_0}(\hat{p} \leq \hat{p}_{\text{obs}}) > \frac{\alpha}{2} \right\}. \quad (19)$$

Its limits are the numerical solutions of the equal-tail equations, which are unique when the tail probabilities are monotone in p_0 . A confidence interval for R can then be formed from the component sets through the monotonicity argument used in **Equation 18**. This procedure is outlined here for completeness and is not part of the numerical comparison.

6. BAYESIAN ESTIMATION

6.1 Priors and Posterior Distributions

For one complete shifted geometric observation, the Fisher information is

$$I(p) = \frac{1}{p^2(1-p)}. \quad (20)$$

The Jeffreys prior is therefore

$$\pi_J(p) \propto p^{-1}(1-p)^{-1/2}, \quad (21)$$

which is formally a $\text{Beta}(0, 1/2)$ density and is improper. For independent parameters the joint prior is the product $\pi_J(p_X)\pi_J(p_Y)$. This prior is derived from the complete-sample information of the model and is combined here with the UHCS likelihood, whose information differs because of the random stopping rule. Combining π_J with the likelihood kernel in **Equation 5** gives

$$p_X \mid \text{data} \sim \text{Beta}\left(H_X, A_X + \frac{1}{2}\right), \quad p_Y \mid \text{data} \sim \text{Beta}\left(H_Y, A_Y + \frac{1}{2}\right). \quad (22)$$

The posteriors are proper if and only if $H_X \geq 1$ and $H_Y \geq 1$, which holds under the UHCS. Under squared-error loss the Bayes estimators are

$$\tilde{p}_X = \frac{H_X}{H_X + A_X + \frac{1}{2}}, \quad \tilde{p}_Y = \frac{H_Y}{H_Y + A_Y + \frac{1}{2}}. \quad (23)$$

For the uniform prior $\text{Beta}(1, 1)$, the posteriors are $\text{Beta}(H_X + 1, A_X + 1)$ and $\text{Beta}(H_Y + 1, A_Y + 1)$ with posterior means

$$\frac{H_X + 1}{H_X + A_X + 2}, \quad \frac{H_Y + 1}{H_Y + A_Y + 2}. \quad (24)$$

The uniform prior is included to examine sensitivity to the prior specification.

6.2 Bayesian Estimation of the Reliability

Under squared-error loss, the Bayes estimator of R is the posterior mean

$$\hat{R}_B = \mathbb{E} \left[\frac{(1 - p_X)p_Y}{p_X + p_Y - p_X p_Y} \mid \text{data} \right], \quad (25)$$

which exists because $R \in (0, 1)$ and is evaluated by Monte Carlo integration. For $s = 1, \dots, S$, independent draws $p_X^{(s)}$ and $p_Y^{(s)}$ are generated from the posteriors, and $R^{(s)} = R(p_X^{(s)}, p_Y^{(s)})$ is computed from **Equation 11**. The estimator is

$$\hat{R}_B = \frac{1}{S} \sum_{s=1}^S R^{(s)}, \quad (26)$$

and an equal-tail $100(1 - \alpha)\%$ credible interval is formed from the empirical $\alpha/2$ and $1 - \alpha/2$ quantiles of $\{R^{(s)}\}$.

7. BOOTSTRAP INTERVALS

The parametric bootstrap interval is obtained as follows:

1. compute $\hat{p}_X$ and $\hat{p}_Y$ from the observed UHCS data;
2. generate complete uncensored samples of sizes n and m from the shifted geometric distributions with parameters $\hat{p}_X$ and $\hat{p}_Y$;
3. apply the same UHCS rule and tie convention to each sample;
4. recompute $\hat{p}_X^b$, $\hat{p}_Y^b$, and $\hat{R}^b$; and
5. repeat for $b = 1, \dots, B$ and take the empirical $\alpha/2$ and $1 - \alpha/2$ quantiles of $\hat{R}^b$.

A posterior-predictive variant draws p_X and p_Y from the posteriors instead of fixing them at the estimates, generates UHCS samples, and recomputes $\hat{R}_B$. Such an interval combines posterior and repeated-sampling variability, so that its interpretation differs from that of a credible interval and of a bootstrap confidence interval. This variant is not evaluated in the present study.

8. SIMULATION DESIGN

The baseline configuration is $p_X = 0.05$ and $p_Y = 0.10$, for which $R = 0.655172$ by **Equation 11**. The sample-size pairs (n, m) are $(250, 250)$, $(250, 400)$, $(400, 400)$, $(400, 600)$, and $(600, 600)$, all of which exceed 200 units per sample. The UHCS constants are $k = \lfloor 0.2n \rfloor$ and $r = \lfloor 0.6n \rfloor$ for the X sample and $k = \lfloor 0.2m \rfloor$ and $r = \lfloor 0.6m \rfloor$ for the Y sample, with $T_1 = 5$ and $T_2 = 20$ for both samples. For these parameter values, simulation shows that about 61 percent of the X units and about 62 percent of the Y units fail before termination. The simulation is a pilot study restricted to this single parameter pair and this single censoring level.

Each replication generates independent X and Y samples, applies **Equation 3** with the tie convention of **Section 2**, computes (H, A) for both samples, and evaluates the three point estimators, namely maximum likelihood, Jeffreys Bayes, and uniform-prior Bayes, and four interval estimators for R , namely delta method, conditional beta Bonferroni, Jeffreys credible, and parametric bootstrap. For an estimator $\hat{\theta}_j$ in replication j of M replications,

$$\text{Bias} = \frac{1}{M} \sum_{j=1}^M (\hat{\theta}_j - \theta), \quad (27)$$

$$\text{MSE} = \frac{1}{M} \sum_{j=1}^M (\hat{\theta}_j - \theta)^2, \quad (28)$$

and for intervals (L_j, U_j) the coverage probability and average interval length are

$$CP = \frac{1}{M} \sum_{j=1}^M \mathbf{1}\{L_j \leq \theta \leq U_j\}, \quad (29)$$

$$AIL = \frac{1}{M} \sum_{j=1}^M (U_j - L_j). \quad (30)$$

The Monte Carlo standard error of a coverage estimate is $\sqrt{CP(1 - CP)/M}$. The point-estimation experiment uses $M = 2000$ replications, and the interval experiment uses $M = 400$ replications, for which the standard error at a nominal coverage of 0.95 is 0.0109.

9. SIMULATION RESULTS

9.1 Point Estimation

Table 1 reports the bias and mean squared error of the estimators of p_X , p_Y , and R . Bias is shown for the three estimators in **Figure 2** for balanced samples, and the mean squared error of the estimators of R is shown in **Figure 3**.

Table 1: Bias and Mean Squared Error of the Point Estimators

n	m	Estimator	Bias ($\times 10^{-3}$)			MSE ($\times 10^{-6}$)		
			$\hat{p}_X$	$\hat{p}_Y$	$\hat{R}$	$\hat{p}_X$	$\hat{p}_Y$	$\hat{R}$
250	250	MLE	0.424	0.488	-1.293	16	59	647
		Jeffreys	0.416	0.455	-1.714	16	59	648
		Uniform	0.722	1.005	-1.923	17	60	647
250	400	MLE	0.327	0.381	-0.797	17	39	594
		Jeffreys	0.319	0.361	-0.879	17	39	591
		Uniform	0.625	0.704	-1.578	17	40	595
400	400	MLE	0.085	0.313	0.069	10	35	410
		Jeffreys	0.079	0.293	-0.194	10	35	409
		Uniform	0.269	0.636	-0.299	10	36	408
400	600	MLE	0.233	-0.010	-1.219	10	25	346
		Jeffreys	0.228	-0.024	-1.322	10	25	345
		Uniform	0.419	0.205	-1.673	10	25	348
600	600	MLE	0.063	0.209	0.032	7	24	276
		Jeffreys	0.059	0.195	-0.152	7	24	277
		Uniform	0.186	0.424	-0.230	7	24	275

Note: Moderate UHCS with $p_X = 0.05$, $p_Y = 0.10$, $T_1 = 5$, $T_2 = 20$, and $M = 2000$. Estimators: maximum likelihood (MLE), Bayes under the geometric Jeffreys prior (Jeffreys), and Bayes under the uniform prior (Uniform).

The mean squared error decreases with sample size for every estimator. For R in balanced samples, the root mean squared error of the maximum likelihood estimator is 0.025, 0.020, and 0.017 at $n = m = 250, 400$, and 600, which follows the $n^{-1/2}$ rate. The three estimators of R have mean squared errors that agree within 2×10^{-6} in the balanced designs, and for unbalanced designs the mean squared error lies between those of the two corresponding balanced designs.

All biases are small relative to the root mean squared error. The absolute bias of $\hat{R}$ does not exceed 1.923×10^{-3} , which is 7.6 percent of the root mean squared error at $n = m = 250$, where the Monte Carlo standard error of the bias is about 0.57×10^{-3} . The maximum likelihood and Jeffreys estimators of p_X and p_Y have biases that differ by less than 0.04×10^{-3} in every design. The uniform prior gives the largest positive bias for p_X and p_Y and the largest negative bias for R in every design, as shown in **Figure 2**. The bias of the Jeffreys estimator of R is more negative than that of the maximum likelihood estimator by 0.42×10^{-3} , 0.26×10^{-3} , and 0.18×10^{-3} at $n = m = 250, 400$, and 600. These differences are smaller than

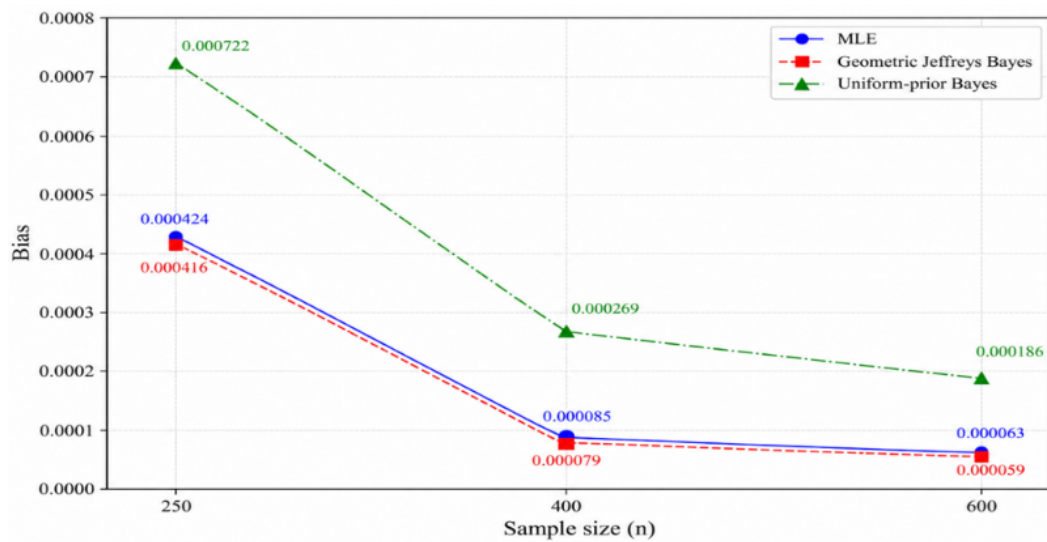
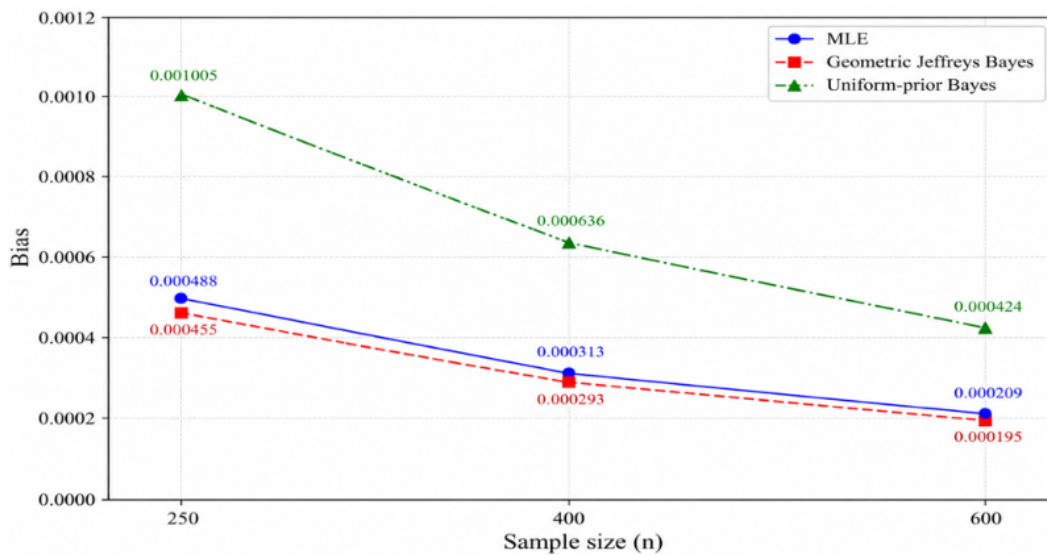
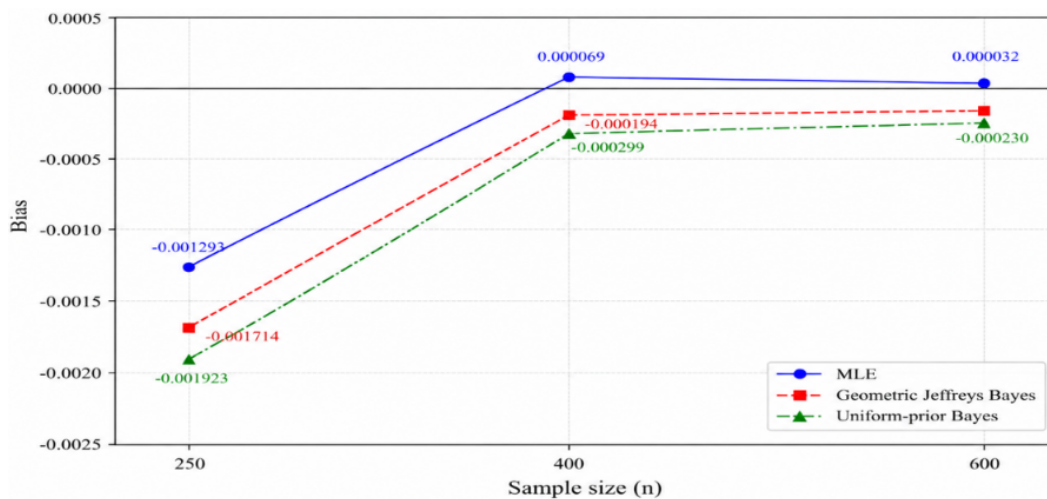

(a) Bias of $\hat{p}_X$

(b) Bias of $\hat{p}_Y$

(c) Bias of $\hat{R}$

Figure 2: Bias of the Estimators for Balanced Samples

the Monte Carlo standard errors of the bias of the maximum likelihood estimator, which are about 0.57×10^{-3} , 0.45×10^{-3} , and 0.37×10^{-3} at the three sizes, and the pilot results do not establish the superiority of any of the three estimators.

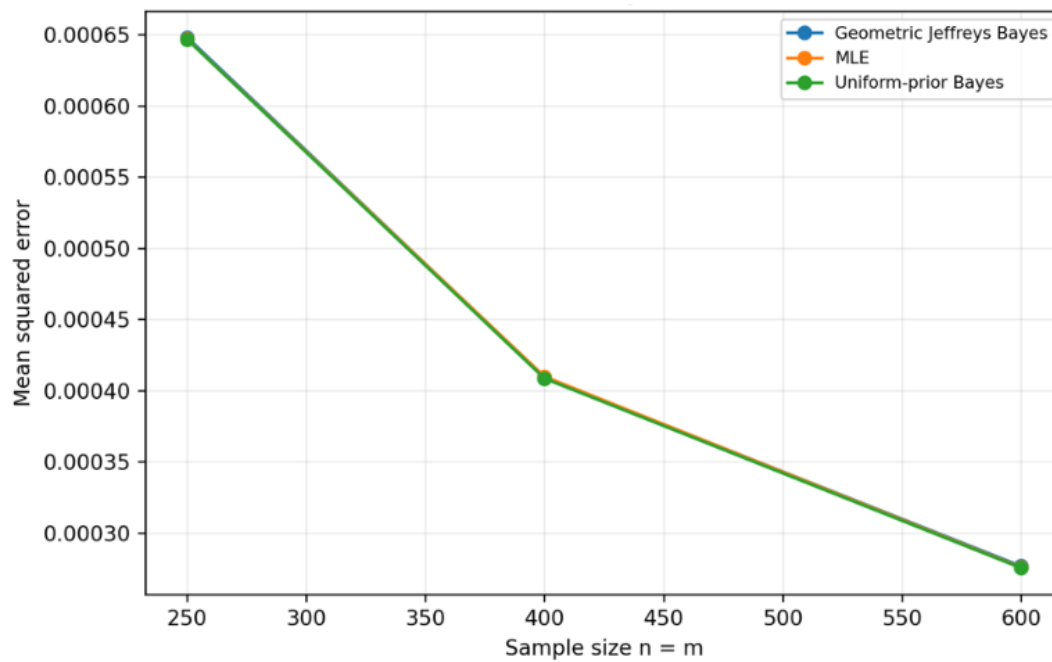


Figure 3: Mean Squared Error of the Estimators of R for Balanced Samples

9.2 Interval Estimation

Table 2 reports the coverage probability and average length of the four 95 percent intervals for R in balanced samples. Coverage is plotted in **Figure 4**, where the dashed horizontal line marks the nominal level, and average length is plotted in **Figure 5**.

Table 2: Coverage Probability and Average Length of 95% Intervals for R

$n = m$	Interval	CP	MC SE	AIL
250	Delta method	0.9675	0.0089	0.0996
	Conditional beta Bonferroni	1.0000	0.0000	0.1627
	Jeffreys credible	0.9600	0.0098	0.0991
	Parametric bootstrap	0.9625	0.0095	0.0990
400	Delta method	0.9550	0.0104	0.0788
	Conditional beta Bonferroni	0.9950	0.0035	0.1286
	Jeffreys credible	0.9550	0.0104	0.0784
	Parametric bootstrap	0.9425	0.0116	0.0776
600	Delta method	0.9475	0.0112	0.0643
	Conditional beta Bonferroni	0.9950	0.0035	0.1049
	Jeffreys credible	0.9450	0.0114	0.0641
	Parametric bootstrap	0.9400	0.0119	0.0636

Note: Moderate UHCS with $p_X = 0.05$, $p_Y = 0.10$, $T_1 = 5$, $T_2 = 20$, and $M = 400$. CP: coverage probability. MC SE: Monte Carlo standard error $\sqrt{CP(1 - CP)/M}$, which equals zero when CP equals one. AIL: average interval length.

The conditional beta Bonferroni interval has coverage of 1.000, 0.995, and 0.995, and it is about 1.6 times longer than the delta-method interval at every sample size, as shown in **Figure 5**. This is the expected conservativeness of a Bonferroni construction, and high coverage alone is not evidence of better performance. The delta-method, Jeffreys credible, and parametric bootstrap intervals have coverage between 0.940 and 0.968, as shown in **Figure 4**, and all of these values lie within 1.96 Monte Carlo standard errors of 0.95, that is, within the interval from 0.929 to 0.971. The lowest values, 0.9425 and 0.940 for the bootstrap interval at $n = m = 400$ and 600, therefore cannot be distinguished from nominal coverage with

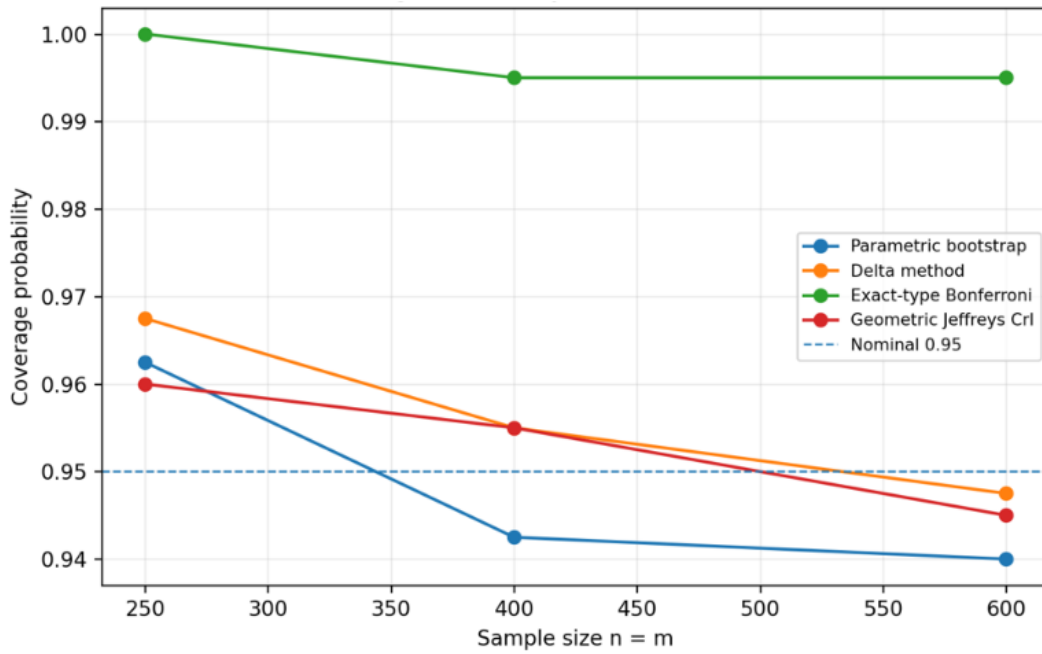


Figure 4: Coverage Probability of 95% Intervals for R

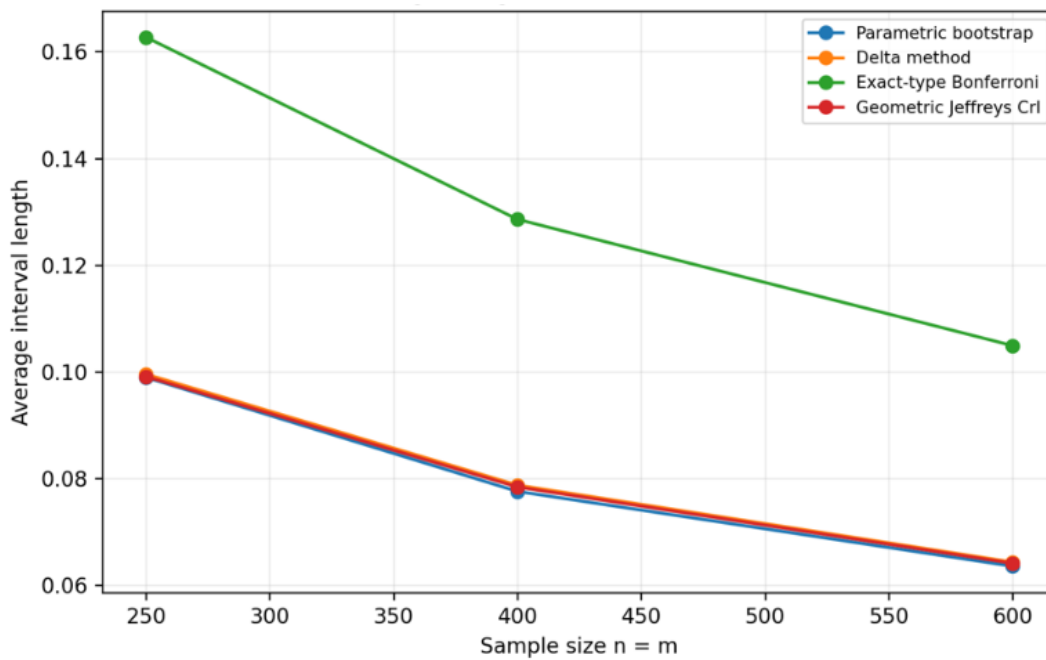


Figure 5: Average Length of 95% Intervals for R

$M = 400$. The three intervals have average lengths within 2 percent of one another at each sample size, with the bootstrap interval shortest. Average length decreases approximately in proportion to $n^{-1/2}$, from about 0.100 at $n = m = 250$ to 0.079 at 400 and 0.064 at 600 for the delta-method interval.

10. APPLICATION TO ACCELERATED CAPACITOR LIFETIMES

The data are accelerated capacitor lifetimes from the reliability data collection distributed with the R package `survival` [23]. Two groups at the same temperature that differ in applied voltage are compared, with the lower-voltage group taken as the X population and the higher-voltage group as the Y population. In this setting $R = \mathbb{P}(X > Y)$ is the probability that a lower-voltage unit outlasts a higher-voltage unit. It is a comparative lifetime measure and not a mechanical strength-versus-load reliability.

Lifetimes $T > 0$ are recorded on a continuous hour scale and are converted to inspection-cycle counts by

$$Z_{\Delta} = \lceil T/\Delta \rceil, \quad (31)$$

where $\Delta > 0$ is the cycle width in hours. The width $\Delta = 100$ hours is adopted as the reference case, and $\Delta \in \{1, 10, 50, 100, 200\}$ is used to examine sensitivity. At $\Delta = 100$ the sufficient statistics are $H_X = 4$, $A_X = 82$, $H_Y = 4$, and $A_Y = 36$. From **Equation 7**, $\hat{p}_X = 4/86 = 0.046512$ and $\hat{p}_Y = 4/40 = 0.100000$, and **Equation 11** gives $\hat{R}_{ML} = 0.672131$. Under the Jeffreys prior, $\tilde{p}_X = 4/86.5 = 0.046243$ and $\tilde{p}_Y = 4/40.5 = 0.098765$. The posterior mean of R is $\hat{R}_B = 0.655$ with a 95 percent credible interval of $(0.319, 0.899)$.

Table 3 and **Table 4** report the results for all cycle widths, and **Figure 6** shows the estimates of R against Δ . The number of observed failures is $H_X = H_Y = 4$ for every Δ , since the failure indicators are fixed by the continuous data before conversion.

Table 3: Parameter Estimates for the Capacitor Data by Cycle Width

Δ (h)	A_X	$\hat{p}_X$	$\tilde{p}_X$	A_Y	$\hat{p}_Y$	$\tilde{p}_Y$
1	7,956	0.000503	0.000502	3,799	0.001052	0.001052
10	796	0.005000	0.004997	378	0.010471	0.010458
50	161	0.024242	0.024169	75	0.050633	0.050314
100	82	0.046512	0.046243	36	0.100000	0.098765
200	40	0.090909	0.089888	17	0.190476	0.186047

Table 4: Reliability Estimates and 95% Jeffreys Credible Intervals for the Capacitor Data by Cycle Width

Δ (h)	$\hat{R}_{ML}$	$\hat{R}_B$	Lower limit	Upper limit
1	0.6766	0.6589	0.3211	0.9023
10	0.6757	0.6582	0.3207	0.9033
50	0.6708	0.6535	0.3149	0.8992
100	0.6721	0.6551	0.3192	0.8990
200	0.6557	0.6401	0.3082	0.8916

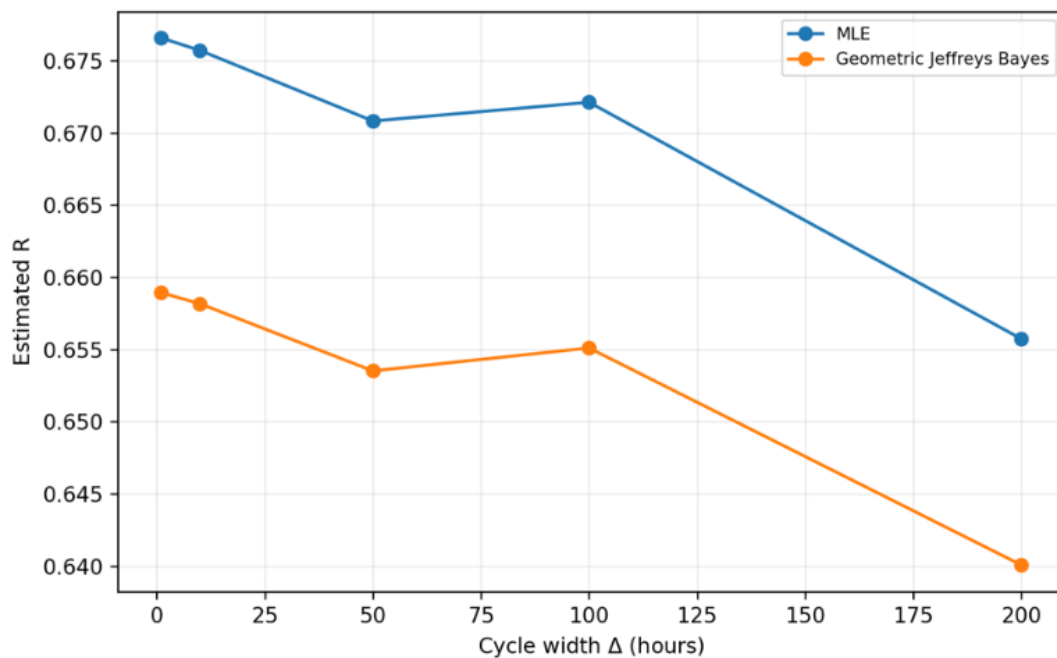


Figure 6: Estimated Reliability of the Capacitor Data versus Cycle Width

The per-cycle parameters change by a factor of about 180 over the range of Δ , as shown in **Table 3**, from $\hat{p}_X = 0.000503$ to 0.090909 and from $\hat{p}_Y = 0.001052$ to 0.190476 . For small Δ , $\hat{p}$ is approximately proportional to Δ , because A is approximately the total observed time divided by Δ . In contrast, $\hat{R}_{ML}$ remains between 0.656 and 0.677 , while $\hat{R}_B$

remains between 0.640 and 0.659, as shown in **Table 4** and **Figure 6**. This stability has a simple explanation under the additional assumption of constant hazard. If the underlying continuous lifetimes are exponential with rates λ_X and λ_Y , then $p_X = 1 - e^{-\lambda_X \Delta}$ and $p_Y = 1 - e^{-\lambda_Y \Delta}$, and R in **Equation 11** tends to $\lambda_Y / (\lambda_X + \lambda_Y)$ as $\Delta \rightarrow 0$, a quantity that does not depend on Δ . At $\Delta = 1$, the ratio $\hat{p}_Y / (\hat{p}_X + \hat{p}_Y) = 0.6767$ is close to $\hat{R}_{ML} = 0.6766$. The credible interval width stays near 0.58 for all Δ . The shift of the estimates at $\Delta = 200$ is the largest, about 0.016 for the maximum likelihood estimate relative to $\Delta = 100$, and is small compared with the width of the credible interval. Because the credible intervals in **Table 4** are wide, the stability of the point estimates in **Figure 6** should be read together with this uncertainty. The constant-hazard assumption was not tested, and four failures per group do not permit a meaningful goodness-of-fit assessment.

The posterior distribution of R at $\Delta = 100$ hours is displayed in **Figure 7**. It is skewed to the left, which is consistent with a posterior mean below the maximum likelihood estimate. The credible interval is wide because only four failures are available in each group, and it contains 0.5, so the data do not establish an ordering of the two groups at the 95 percent level. Given the small number of observed failures, the capacitor analysis is an illustration of computational feasibility and of sensitivity to the cycle width, and not a definitive reliability comparison of the two voltage levels.

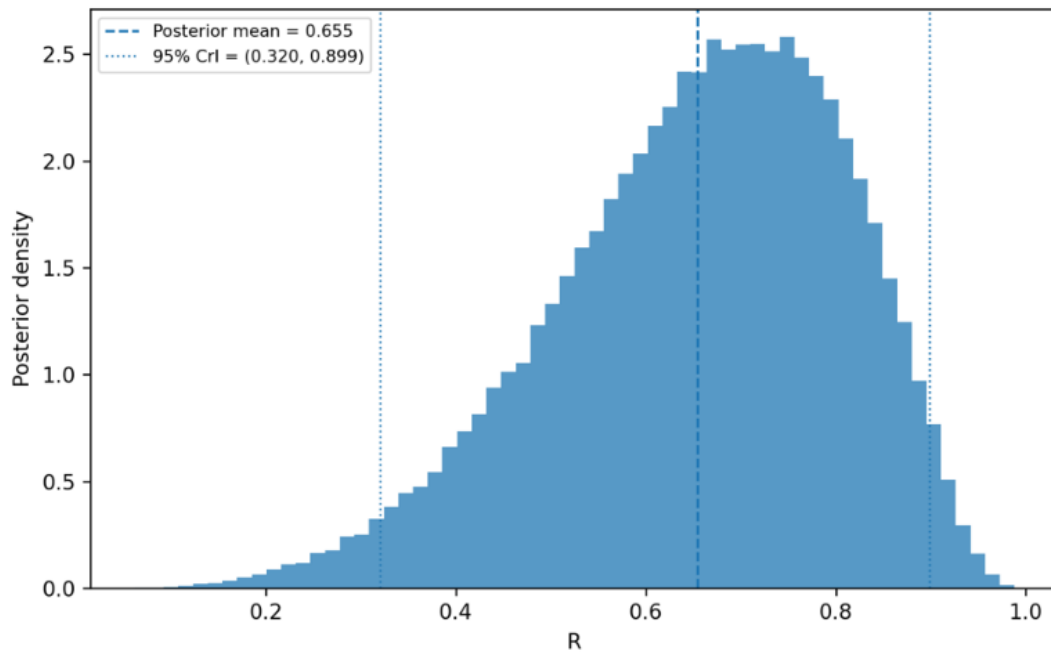


Figure 7: Posterior Distribution of R for the Capacitor Data at $\Delta = 100$ Hours

11. DISCUSSION

The shifted geometric model has a tractable structure under UHCS. Once a censored sample is summarized by the failure count H and the total exposure A , the likelihood kernel $p^H(1-p)^A$ gives closed-form maximum likelihood estimators and conjugate beta posteriors. The maximum likelihood and Jeffreys estimators, $H/(H+A)$ and $H/(H+A+\frac{1}{2})$, differ by a factor $1 + 1/\{2(H+A)\}$ and are therefore nearly equal when $H+A$ is large, which explains the close agreement reported in **Section 9**. The uniform prior adds one to both H and A , and its shrinkage is more visible when H is small.

Interval estimation shows a trade-off between coverage and length. The conditional beta Bonferroni interval attains very high coverage at the price of intervals about 1.6 times longer, whereas the delta-method, Jeffreys credible, and bootstrap intervals have coverage consistent with 0.95 and much shorter lengths in the pilot study. The deviations of the bootstrap coverage from 0.95 are within Monte Carlo error at the replication level used.

The capacitor analysis illustrates a modeling point. The geometric parameter depends on the definition of one cycle, so that its numerical value has no meaning independent of Δ , whereas R is nearly invariant to the cycle width.

Several limitations apply. The simulation is a pilot study with one parameter pair, one censoring level, and moderate replication counts, so that conclusions about other parameter values and censoring levels are not supported. The exact test-inversion procedure and the posterior-predictive bootstrap interval are described but not evaluated. The coverage of the conditional beta interval under the full adaptive stopping rule rests on simulation only. The real-data illustration uses a single data set with four failures per group, and the constant-hazard assumption underlying the geometric model was not tested.

12. CONCLUSION

Inference for the stress-strength reliability $R = \mathbb{P}(X > Y)$ of independent shifted geometric lifetimes under UHCS has been developed. The established analytical results are the closed-form maximum likelihood estimators of p_X , p_Y , and R , the Wald and delta-method intervals, the conservative conditional beta interval, and the beta posteriors under the Jeffreys and uniform priors with Monte Carlo evaluation of the posterior of R . The pilot simulation indicates that the maximum likelihood and Jeffreys estimators perform almost identically for sample sizes between 250 and 600, and that the delta-method, Jeffreys credible, and bootstrap intervals attain coverage consistent with the nominal level with intervals about 39 percent shorter than the conditional beta interval. In the capacitor data, R is far more stable than the per-cycle parameters when the cycle width varies from 1 to 200 hours, although the constant-hazard assumption was not tested and only four failures per group are available.

Future work includes implementing the exact test-inversion interval, extending the simulation to further parameter values and to heavy and light censoring, and evaluating the posterior-predictive bootstrap interval. Other directions are discrete lifetime models with nonconstant hazard, dependent stress and strength, and progressive censoring schemes.

AUTHOR CONTRIBUTION STATEMENT

All authors contributed equally to the study conception and design. Material preparation, data collection, and analysis were performed by the authors. The first draft of the manuscript was written by the authors, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

ETHICS APPROVAL AND CONSENT TO PARTICIPATE

Ethics declaration: not applicable. This study did not involve human participants or animals.

CONSENT FOR PUBLICATION

Not Applicable.

DATA AVAILABILITY

The accelerated capacitor lifetime data used in the real-data illustration are publicly available through the reliability-data collection distributed with the `survival` package in R. The simulation data were generated by the authors using the parameter settings, censoring configurations, and random-generation procedures described in the manuscript.

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The authors declare no conflict of interest.

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