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Mixed-Record Stress-Strength Reliability for the Poisson-Exponential Distribution

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ABSTRACT

This study develops classical and Bayesian inference for the stress–strength reliability parameter R=P(X>Y) under an asymmetric mixed-record sampling scheme for the Poisson-exponential distribution. The strength variable is observed through upper record values, whereas the stress variable is represented by lower record values. Exact likelihood contributions are derived from the two record processes and combined under independence. To improve numerical stability when only short record sequences are available, the main inferential formulation assumes a commonly known or externally calibrated rate parameter, while the two shape parameters are estimated from their corresponding records. Maximum likelihood estimation is developed together with a logit-based asymptotic confidence interval using the observed information and the delta method. Bayesian inference is formulated under independent flat working priors, and a parametric bootstrap-Bayesian percentile interval is also considered. A Monte Carlo study evaluates bias, RMSE, empirical coverage probability, and average interval length under low-, moderate-, and high-reliability scenarios. The Bayesian posterior-mean estimator yields lower RMSE than the MLE across all reported settings, while Bayesian intervals generally provide coverage closer to the nominal level and shorter average lengths. The proposed methodology is further illustrated using published steel-specimen lifetime data, demonstrating both the feasibility and uncertainty of reliability inference based on limited mixed-record observations.

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1. INTRODUCTION

Stress–strength reliability is an essential concept in reliability theory that measures a system's ability to withstand an applied load. Let X be the strength of a system and Y be the stress on it [1]. The corresponding measure of reliability is given by R=P(X>Y), the probability that a randomly selected strength exceeds a randomly selected stress. This formulation has been widely used in reliability engineering, lifetime analysis, quality control, and material-performance assessment, where the system performance is determined by the interaction between resistance and external loading[2].

The choice of proper lifetime distribution is an important aspect of stress–strength analysis. Among the existing models, the Poisson-Exponential (PE) distribution proposed by [3] is a two-parameter extension of the exponential lifetime distribution obtained from a complementary-risk formulation. The model possesses an increasing failure rate and admits explicit expressions for several reliability-related quantities.[4] later developed Bayesian inference for the PE distribution, including estimation of its parameters and reliability characteristics.

However, in many reliability experiments, the complete sequence of observations may not be available, and only successive extreme observations are retained. These observations are called record values. An upper record is defined as a new observation that exceeds all previous observations, and a lower record is defined as a new observation that is less than all previous observations. The statistical foundation of the theory of record values was established by [5] and further developed in a systematic manner by [6]. Record samples contain only successive extremes and thus represent a highly reduced form of the original data, requiring inferential procedures specially adapted to the information structure.

Stress-strength reliability based on lower record values has been studied for several lifetime models.[7] proposed likelihood and Bayesian estimators. The attainment of upper-record stress-strength reliability has also drawn considerable attention. [8] proposed the maximum likelihood and Bayesian procedures for P(X>Y) using upper records from the Kumaraswamy distribution. [9] studied stress-strength reliability for a two-parameter bathtub-shaped life distribution under upper record sampling and developed classical, Bayesian, and bootstrap interval methods. It also included inter-record times in the Weibull stress-strength reliability analysis. Thus, they used additional information other than just the record values. More recently, [10] considered maximum likelihood, Bayesian, bootstrap, and posterior interval procedures for stress-strength reliability based on upper record values for the inverted exponentiated Pareto distribution.

Therefore, the existing literature provides well-developed inferential procedures when the two stress-strength samples are observed under a common record direction. Specifically, the lower-record studies mentioned above [11], [12] are constructed based on lower-record information, while the upper-record studies [8], [9], [10] are constructed based on upper records. But the asymmetric case where lower records represent the stress process and upper records represent the strength process has received comparatively less attention. This mixed configuration is statistically different, as the contribution to the likelihood of the lower records depends on the cumulative distribution function, while the contribution of the upper records depends on the survival function. Hence, an inferential approach that combines the two record directions in a particular way is required.

Motivated by this gap, the present paper proposes a mixed-record stress-strength reliability for the Poisson-exponential distribution where the strength variable is measured by upper record values, and the stress variable is measured by lower record values. We calculate exact likelihood contributions for the two record processes and combine them with assuming independence. While the general PE model has shape and rate parameters for each population, estimation of all unknown parameters from short record sequences can lead to weakly identified or poorly conditioned likelihoods. The main inferential development thus assumes a known or externally calibrated common rate parameter and estimates the two PE shape parameters from the respective record sequences. This restriction is made explicit and provides a numerically stable foundation for mixed-record inference.

The main contribution of this study is the integration of the PE stress–strength model and an asymmetric lower-stress/upper-strength record mechanism into a single classical and Bayesian setting. Specifically, the exact mixed-record likelihood is derived, maximum likelihood estimation is developed under the common calibrated-rate formulation, the asymptotic confidence interval is constructed using the observed information, delta method, and logit transformation, Bayesian inference is formulated under independent flat working priors, and a parametric bootstrap-

Bayesian percentile interval is considered for the posterior-mean reliability estimator. The finite-sample performance of the proposed procedures is evaluated via Monte Carlo simulation in terms of bias, RMSE, empirical coverage probability, and average interval length under low, moderate, and high reliability settings. The methodology is illustrated by utilizing published lifetime observations on steel specimens.

The rest of the paper is structured as follows. In Section 2, we introduce the Poisson-exponential distribution and study its main distributional properties. Section 3 develops the stress-strength reliability model and the mixed lower-upper record sampling scheme. Section 4 develops maximum likelihood, asymptotic, Bayesian, and bootstrap-Bayesian inference. Section 5 describes the Monte Carlo simulation study. Section 6 presents the real-data application, record extraction, parameter calibration, and model assessment. Section 7 presents the numerical results based on the records; Section 8 discusses the main findings and limitations; and Section 9 concludes the study.

2. THE POISSON-EXPONENTIAL DISTRIBUTION

The Poisson–Exponential distribution provides a flexible extension of the exponential model through an additional shape parameter. In the general stress–strength setting, $X \sim PE(\theta_X, \beta_X)$and $Y \sim PE(\theta_Y, \beta_Y)$. However, record-only likelihoods may be numerically unstable when both shape and rate parameters are estimated from short record sequences. Therefore, this study adopts the common-rate model

$$X \sim PE(\theta_X, \beta),\ Y \sim PE(\theta_Y, \beta),$$

where $\beta$is assumed known or externally calibrated. This assumption improves numerical stability and leads to a closed-form expression for the reliability measure Rthat is independent of β.

$$X \sim PE(\theta_X, \beta_X), \qquad Y \sim PE(\theta_Y, \beta_Y),$$

The exact mixed-record likelihood can be written directly. However, numerical profiling of record-only likelihoods can be poorly conditioned when both the shape and rate parameters are estimated separately from each short record sequence. Boundary solutions may occur because record values contain much less information than complete samples. Therefore, the inferential and simulation sections of this paper use

$$X \sim PE(\theta_X, \beta), \qquad Y \sim PE(\theta_Y, \beta),$$

with a commonly known or externally calibrated rate β. This assumption is transparent and is similar in spirit to record-based stress-strength analyses that treat a scale parameter as known. It also has the important advantage that R has a closed form independent of β. A fully unknown three- or four-parameter record model should preferably use additional information, such as record times, inter-record waiting counts, or external calibration data.
For more statistical information on the Poisson-Exponential Distribution (PED);
Let $Z \sim PE(\theta, \beta)$ with $\theta > 0$ and $\beta > 0$. Its cumulative distribution function (CDF) is

$$F(z; \theta, \beta) = \frac{\exp\{-\theta e^{-\beta z}\} - e^{-\theta}}{1 - e^{-\theta}}, \qquad z > 0.$$

The survival function is

$$S(z; \theta, \beta) = \frac{1 - \exp\{-\theta e^{-\beta z}\}}{1 - e^{-\theta}},$$

and the probability density function (PDF) is

$$f(z; \theta, \beta) = \frac{\theta \beta e^{-\beta z} \exp\{-\theta e^{-\beta z}\}}{1 - e^{-\theta}}.$$

As $\theta \to 0$, the PE distribution converges to the exponential distribution with rate β.
The inverse CDF is useful for simulation. If $U \sim U(0,1)$, then

$$Z = -\frac{1}{\beta} \log\left[-\frac{1}{\theta} \log\{e^{-\theta} + U(1 - e^{-\theta})\}\right].$$

3. THE STRESS-STRENGTH MODEL OF RELIABILITY

In the stress–strength reliability framework, let $X \sim PE(\theta_X, \beta)$,and $Y \sim PE(\theta_Y, \beta)$be independent Poisson–Exponential random variables with shape parameters $\theta_X$and $\theta_Y$and a common rate parameter β.

The reliability measure is defined as

$$R = P(Y < X) = \int_0^\infty F_Y(x) f_X(x)\, dx.$$

For the common-rate PE model, the reliability has the closed form

$$R(\theta_X, \theta_Y) = \frac{\theta_X}{(e^{\theta_X}-1)(e^{\theta_Y}-1)}\left[\frac{e^{\theta_X+\theta_Y-1}}{\theta_X+\theta_Y} - \frac{e^{\theta_X-1}}{\theta_X}\right]$$

Thus, R depends on θ_X and θ_Y but not on the common rate β.

3.1. Mixed Record Sampling Scheme

Record-value data provides an efficient framework for reliability inference when only successive extremes are observed. Let $X_1^U < \cdots < X_n^U$ denote the upper records from the strength process and $Y_1^L > \cdots > Y_m^L$the lower records from the stress process. Using the standard likelihoods for upper and lower records, and assuming independence between the two processes. For a continuous distribution with CDF F and PDF f, the joint likelihood of upper records is

$$L_U \propto f(x_n^U) \prod_{j=1}^{n-1} \frac{f(x_j^U)}{1 - F(x_j^U)},$$

whereas the joint likelihood of lower records is

$$L_L \propto f(y_m^L) \prod_{i=1}^{m-1} \frac{f(y_i^L)}{F(y_i^L)}.$$

Because the two record processes are independent,

$$L(\eta) = L_U L_L.$$

Figure 1 provides a summary of the overall methodological framework adopted for stress–strength reliability inference under the proposed asymmetric mixed-record scheme. We start with two independent Poisson-exponential processes for strength and stress, and we keep upper record values for the strength process and lower record values for the stress process, keeping the original order of observations. These record sequences are incorporated in the Poisson-Exponential stress-strength formulation with a commonly known or externally calibrated rate parameter, and their corresponding upper and lower record likelihood contributions are combined to construct the exact mixed-record likelihood. Based on this likelihood, maximum likelihood estimation, asymptotic interval estimation, Bayesian inference, and a bootstrap-Bayesian percentile procedure are developed. The derived estimators and interval procedures are then studied by Monte Carlo simulation and also illustrated on the lifetime data of steel specimens.

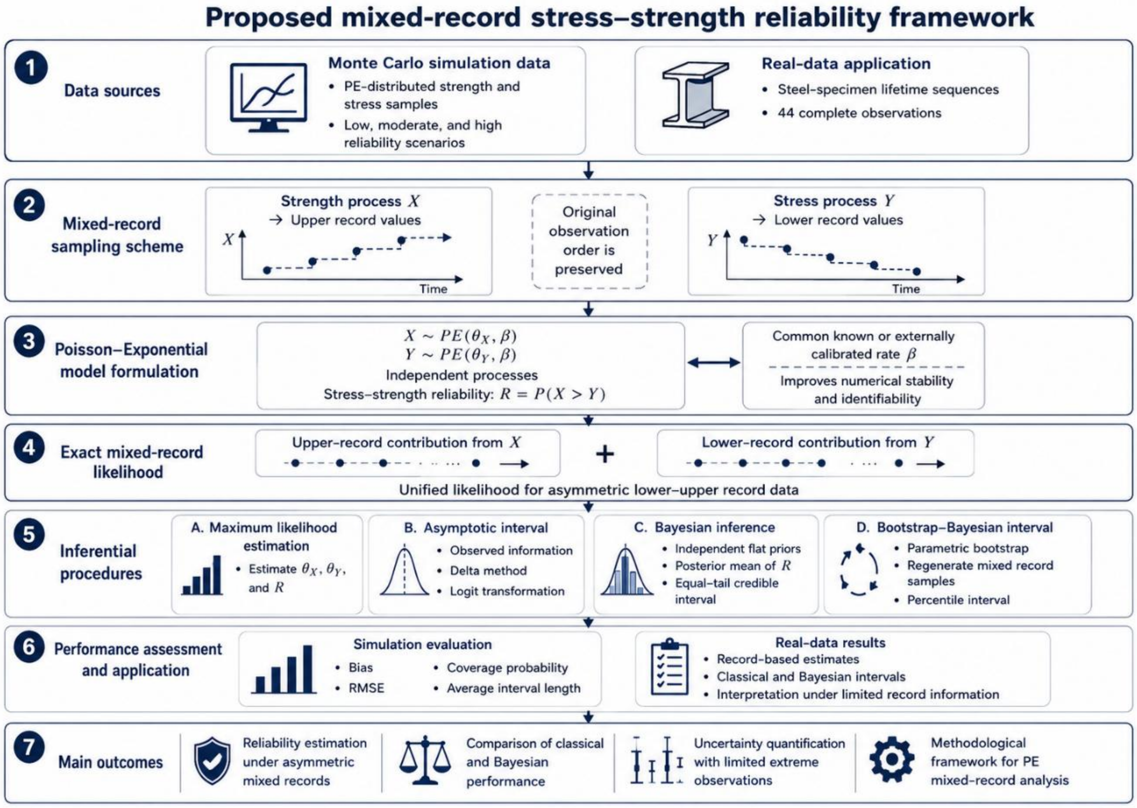

Figure 1: Proposed Mixed-Record Stress–Strength Reliability Framework

4. Inferences for mixed model of stress-strength reliability

4.1. Maximum Likelihood Estimation under a Known Common Rate

To obtain maximum likelihood estimates under the mixed-record setting, the exact upper- and lower-record likelihood contributions are first derived for the general Poisson–Exponential model. Under the commonly known rate assumption, $\beta_X = \beta_Y = \beta$, inference reduces to estimating the two shape parameters separately from their corresponding record sequences. The resulting log-likelihoods and score equations are solved numerically to obtain $\hat{\theta}_X$and $\hat{\theta}_Y$, after which the maximum likelihood estimator of the stress–strength reliability is obtained by invariance as

$$\hat{R}_{ML} = R(\hat{\theta}_X, \hat{\theta}_Y).$$

$$u_j = e^{-\beta_X x_j^U}.$$

$$L_U(\theta_X, \beta_X) \propto \frac{(\theta_X \beta_X)^n \exp\left(-\beta_X \sum_{j=1}^{n} x_j^U\right) \exp(-\theta_X u_n)}{1 - e^{-\theta_X}} \prod_{j=1}^{n-1} \frac{1}{e^{\theta_X u_j} - 1}.$$

For the stress lower records, define

$$v_i = e^{-\beta_Y y_i^L}.$$

$$L_L(\theta_Y, \beta_Y) \propto \frac{(\theta_Y \beta_Y)^m \exp(-\beta_Y \sum_{i=1}^{m} y_i^L) \exp(-\theta_Y v_m)}{1 - e^{-\theta_Y}} \prod_{i=1}^{m-1} \left[1 - e^{-\theta_Y (1 - v_i)}\right]^{-1}.$$

The equations above represent the exact mixed-record likelihood contributions under the general PE model.

set

$$\beta_X = \beta_Y = \beta,$$

where β is known or externally calibrated.

Ignoring terms that do not involve θ_X, the upper-record log-likelihood

$$\ell_X(\theta_X) = n\log\theta_X - \theta_X u_n - \log\left(1 - e^{-\theta_X}\right) - \sum_{j=1}^{n-1} \log\left(e^{\theta_X u_j} - 1\right) + C_X,$$

where $u_j = e^{-\beta x_j^U}$.

The corresponding score equation is

$$U_X(\theta_X) = \frac{n}{\theta_X} - u_n - \frac{1}{e^{\theta_X} - 1} - \sum_{j=1}^{n-1} \frac{u_j}{1 - e^{-\theta_X u_j}} = 0.$$

For the lower stress records,

$$\ell_Y(\theta_Y) = m\log\theta_Y - \theta_Y v_m - \log\left(1 - e^{-\theta_Y}\right) - \sum_{i=1}^{m-1} \log\left[1 - e^{-\theta_Y(1-v_i)}\right] + C_Y,$$

where $v_i = e^{-\beta y_i^L}$.

The score equation is

$$U_Y(\theta_Y) = \frac{m}{\theta_Y} - v_m - \frac{1}{e^{\theta_Y} - 1} - \sum_{i=1}^{m-1} \frac{1-v_i}{e^{\theta_Y(1-v_i)} - 1} = 0.$$

The MLEs $\hat{\theta}_X$ and $\hat{\theta}_Y$ are obtained by bound one-dimensional numerical maximization

By the invariance property of maximum likelihood,

$$\hat{R}_{ML} = R\left(\hat{\theta}_X, \hat{\theta}_Y\right).$$

4.2. Sensitivity to the calibrated rate parameter

Although the reliability function does not explicitly depend on the common rate parameter β, its estimator is indirectly affected by the assumed value of β. This occurs because the record likelihoods depend on

$$u_j = e^{-\beta x_j^U} \text{ and } v_i = e^{-\beta y_i^L},$$

so that the estimates $\hat{\theta}_X$and $\hat{\theta}_Y$vary with β. Hence,

$$\hat{R}(\beta) = R\left(\hat{\theta}_X, (\beta)\hat{\theta}_Y(\beta)\right)$$

is not generally invariant to the calibration of β.

A sensitivity formulation is considered to characterize the potential effect of misspecification of the calibrated rate parameter around the calibrated value $\beta_0 = 0.00089173$. For each perturbed value of β, the upper- and lower-record models are refitted, and the maximum likelihood and Bayesian estimate of R, together with their associated intervals, are recomputed. The resulting changes in $\hat{R}$quantify the practical effect of possible misspecification of the externally calibrated rate parameter.

For a local analytical assessment, the sensitivity may also be expressed as

$$\frac{d\hat{R}}{d\beta} = \frac{\partial R}{\partial \theta_X}\frac{d\hat{\theta}_X}{d\beta} + \frac{\partial R}{\partial \theta_Y}\frac{d\hat{\theta}_Y}{d\beta}.$$

Using the score equations, the derivatives of the estimators can be obtained implicitly as

$$\frac{d\hat{\theta}_X}{d\beta} = -\left[\frac{\partial U_X}{\partial \theta_X}\right]^{-1}\frac{\partial U_X}{\partial \beta},\ \frac{d\hat{\theta}_Y}{d\beta} = -\left[\frac{\partial U_Y}{\partial \theta_Y}\right]^{-1}\frac{\partial U_Y}{\partial \beta},$$

evaluated at the fitted parameter values. This formulation clarifies that the sensitivity arises through parameter estimation rather than through the reliability function itself.

4.3. Asymptotic Confidence Interval

To quantify the uncertainty in the maximum likelihood estimator of the stress–strength reliability, the observed information matrix is used together with the delta method. Since the strength and stress record sequences are independent and the common rate $\beta$is fixed, the information matrix is diagonal. The resulting variance estimate yields the standard error of $\hat{R}_{ML}$. To ensure that the confidence limits remain within the admissible interval $(0, 1)$, an asymptotic confidence interval is constructed on the logit scale and then transformed back to the reliability scale.

Because the strength and stress record samples are independent and β is treated as fixed, the observed information matrix is diagonal:

$$\mathcal{J}(\theta) = \begin{pmatrix} I_X(\theta_X) & 0 \\ 0 & I_Y(\theta_Y) \end{pmatrix}, \qquad \theta = (\theta_X, \theta_Y)^T.$$

The observed information elements may be evaluated from

$$I_X(\theta_X) = -\frac{\partial^2 \ell_X}{\partial \theta_X^2}, \qquad I_Y(\theta_Y) = -\frac{\partial^2 \ell_Y}{\partial \theta_Y^2},$$

either analytically or by stable numerical differentiation at the MLEs.

Let

$$g(\theta) = \left(\frac{\partial R}{\partial \theta_X}, \frac{\partial R}{\partial \theta_Y}\right)^T.$$

The delta-method variance estimator is

$$\widehat{\text{Var}}(\hat{R}_{ML}) = g(\hat{\theta})^T \mathcal{J}^{-1}(\hat{\theta}) g(\hat{\theta}),$$

and

$$SE(\hat{R}_{ML}) = \sqrt{\widehat{\text{Var}}(\hat{R}_{ML})}.$$

A direct Wald interval can extend outside $(0,1)$. Therefore, define

$$\text{logit}(R) = \log\left(\frac{R}{1-R}\right).$$

By the delta method,

$$SE\{\text{logit}(\hat{R}_{ML})\} = \frac{SE(\hat{R}_{ML})}{\hat{R}_{ML}(1-\hat{R}_{ML})}.$$

A nominal $100(1-\alpha)\%$ asymptotic confidence interval is

$$CI_A = \text{logit}^{-1}\left[\text{logit}\left(\hat{R}_{ML}\right) \pm z_{1-\alpha/2} SE\{\text{logit}\left(\hat{R}_{ML}\right)\}\right]$$

This transformation guarantees endpoints in $(0,1)$.

4.4. Bayesian Inference under a Non-Informative Prior

A Bayesian approach is also considered for estimating the stress–strength reliability under the commonly known rate assumption. Using independent flat priors for $\theta_X$and θ_Y, the joint posterior distribution factorizes into two one-dimensional marginal posteriors corresponding to the upper and lower record samples. Posterior draws of the shape parameters are transformed through $R(\theta_X, \theta_Y)$to obtain the posterior distribution of the reliability measure. The Bayes estimate is then calculated as the posterior mean, while uncertainty is summarized by an equal-tail credible interval.

With β fixed, use independent flat working prior

$$\pi(\theta_X, \theta_Y) \propto 1, \qquad \theta_X > 0,\ \theta_Y > 0.$$

The posterior factorizes:

$$\pi(\theta_X, \theta_Y \mid \mathrm{x}^U, \mathrm{y}^L) \propto L_U(\theta_X \mid x^U) L_L(\theta_Y \mid \mathrm{y}^L).$$

Hence

$$\pi(\theta_X \mid \mathrm{x}^U) \propto L_U(\theta_X), \qquad \pi(\theta_Y \mid \mathrm{y}^L) \propto L_L(\theta_Y).$$

The densities are one-dimensional and can be normalized accurately by quadrature on the log-parameter scale. Posterior samples are then generated by inverse-CDF interpolation.

For posterior draws

$$\theta_X^{(s)}, \quad \theta_Y^{(s)}, \qquad s = 1, \dots, S,$$

Computing

$$R^{(s)} = R\left(\theta_X^{(s)}, \theta_Y^{(s)}\right).$$

Under squared-error loss, the Bayes estimator is the posterior mean

$$\hat{R}_B = E(R \mid \text{records}) \approx \frac{1}{S}\sum_{s=1}^{S} R^{(s)}$$

A 95% equal-tail Bayesian credible interval is

$$CI_B = (R_{0.025}, R_{0.975})$$

where the endpoints are the corresponding empirical posterior quantiles.

To assess sensitivity to the prior specification, the real-data analysis was repeated using independent weakly informative Gamma priors,

$$\theta_X, \theta_Y \sim \text{Gamma}(1, 0.1),$$

where $0.1$is the rate parameter. Under the flat prior, the posterior mean reliability was 0.4943, with a 95%equal-tail credible interval of $(0.1476, 0.8439)$. Under the Gamma prior, the corresponding posterior mean was 0.4711, with credible interval $(0.1483, 0.8264)$. Thus, the posterior mean of Rchanged by approximately 4.7%, while the two credible intervals overlapped substantially. The individual shape-parameter posterior means were more sensitive, particularly for the strength distribution, which is expected because only three upper-strength records were observed. Overall, the reliability conclusion is reasonably stable under the considered prior specifications, although some prior influence on the shape parameters remains.

4.5. Bootstrap-Bayesian Confidence Interval

The bootstrap-Bayesian interval used here is a parametric bootstrap percentile interval for the Bayesian posterior-mean estimator. It is not the same as the posterior credible interval.

Let n be the number of upper strength records and m the number of lower stress records.

The algorithm is:

1. Fit the Bayesian model to the observed records and obtain posterior means $\bar{\theta}_X$ and $\bar{\theta}_Y$.
2. For $b = 1, \ldots, B$, generate an upper-strength record sequence of length n from $PE(\bar{\theta}_X, \beta)$.
3. Generate a lower-stress record sequence of length independently m from $PE(\bar{\theta}_Y, \beta)$.
4. Refit the Bayesian model to the bootstrap record sequences.
5. Compute the bootstrap Bayesian posterior-mean reliability estimate $\hat{R}_B^{*(b)}$.
6. Let $q_{0.025}^*$ and $q_{0.975}^*$ denote the empirical bootstrap quantiles.

$$CI_{BB} = (q_{0.025}^*, q_{0.975}^*).$$

Record sequences can be simulated without generating extremely long raw sequences. If $H(z) = -\log S(z)$, successive upper-record values satisfy exponential increments on the H scale. Similarly, if $G(z) = -\log F(z)$, successive lower-record values satisfy exponential increments on the G scale. This produces the exact conditional record process for continuous distributions.

5. Monte Carlo Simulation Study

The simulation evaluates finite-record performance under a known common rate $\beta = 1$. Three parameter settings were selected to represent low, moderate, and high reliability, as shown in **Table 1**.

Table 1: Parameter settings and corresponding true stress–strength reliability values for the Monte Carlo simulation.

Scenario	θ_X	θ_Y	β	True R
A-low	0.5	3.0	1.0	0.3182
B-mid	1.0	1.0	1.0	0.5000
C-high	3.0	0.5	1.0	0.6818

For each scenario, equal numbers of lower stress and upper strength records were used:

$$m = n \in \{5,10,20\}.$$

For each design point, 200 Monte Carlo replications were generated.

For each replication:

1. Generate n upper strength records from $PE(\theta_X, \beta)$.
2. Generate m lower stress records from $PE(\theta_Y, \beta)$.

3. Compute $\hat{\theta}_X$, $\hat{\theta}_Y$, and $\hat{R}_{ML}$.
4. Construct the 95% logit asymptotic confidence interval.
5. Evaluate the two one-dimensional posterior distributions under the flat prior.
6. Generate posterior draws and compute $\hat{R}_B$ and its 95% equal-tail credible interval.
7. Record estimator bias and squared error.
8. Record interval coverage and interval length.

The reported criteria are

$$\mathrm{Bias}(\hat{R}) = \frac{1}{M}\sum_{b=1}^{M}(\hat{R}_b - R),$$

and

$$\mathrm{RMSE}(\hat{R}) = \sqrt{\frac{1}{M}\sum_{b=1}^{M}(\hat{R}_b - R)^2}$$

5.1 Point-estimation results

Table 2 summarizes the finite-record performance of the maximum likelihood estimator (MLE) and the Bayesian posterior-mean estimator in terms of bias and RMSE (root mean squared error) across the considered low-, moderate-, and high-reliability scenarios. The results reveal that the performance of both estimators is dependent on the underlying reliability level and the number of observed records.

In the low-reliability case (A-low), both estimators are positively biased for all record sizes reported. The Bayesian estimator has a slightly larger bias than the MLE, but it always produces a smaller RMSE. For instance, for five records, the RMSE reduces from 0.2987 for the MLE to 0.2818 for the Bayesian estimator, and for ten records it reduces from 0.2651 to 0.2561. For twenty records, we see a similar pattern, suggesting that posterior averaging provides some reduction in overall estimation error despite the remaining positive bias.

In the moderate-reliability scenario (B-mid), the biases of the two estimators are relatively similar. But the Bayesian estimator still has a lower RMSE at each reported design point. The most obvious improvement in this setting is the MLE and Bayesian RMSE with 30 records, which are 0.2092 and 0.1716, respectively. The results show that the Bayesian estimator is more stable in estimating the reliability parameter under the moderate-reliability setting.

For the high-reliability scenario (C-high), the biases are the lowest of the three scenarios. The bias of the MLE remains close to zero, between 0.0076 and 0.0245. The bias of the Bayesian estimator oscillates slightly around zero, being negative for several record sizes. Despite these small differences in bias, the Bayesian estimator again has a lower RMSE than the MLE for all of the record sizes reported. For example, at 30 records the RMSE decreases from 0.1916 for the MLE to 0.1567 for the Bayesian estimator.

Overall, the Bayesian posterior-mean estimator has smaller RMSE than the MLE for all reported simulation design points, suggesting a better overall point-estimation accuracy under the investigated settings. However, the relation between the number of records and estimation accuracy is not strictly monotonic. In particular, adding more records does not necessarily lead to a commensurate decrease in bias or RMSE. This behavior can be understood from the perspective of record-based inference, since two consecutive record values are forcefully selected extreme observations and do not contain the same information as additional independent and identically distributed observations. Thus, the results must be interpreted in terms of finite-record performance rather than the traditional complete-sample asymptotic.

Table 2: Bias and RMSE of the maximum likelihood and Bayesian estimators under the Monte Carlo simulation scenarios.

Scenario	Records	MLE Bias	MLE RMSE	Bayes Bias	Bayes RMSE
A-low	5	0.1709	0.2987	0.2035	0.2818

A-low	10	0.1454	0.2651	0.1856	0.2561
A-low	20	0.1551	0.2810	0.1940	0.2703
B-mid	5	0.0922	0.2393	0.0906	0.1960
B-mid	10	0.1136	0.2477	0.1091	0.2096
B-mid	20	0.1249	0.2456	0.1127	0.2083
B-mid	30	0.0594	0.2092	0.0640	0.1716
C-high	5	0.0082	0.2037	-0.0101	0.1646
C-high	10	0.0245	0.2202	0.0058	0.1757
C-high	20	0.0076	0.1983	-0.0117	0.1623
C-high	30	0.0081	0.1916	-0.0180	0.1567

Figure 2 Monte Carlo bias of the maximum likelihood and Bayesian posterior-mean estimators for different record sizes in low, moderate, and high reliability scenarios. The bias is positive and largest in the low-reliability scenario. For the high-reliability scenario, the bias is close to zero, especially for the Bayesian estimator.

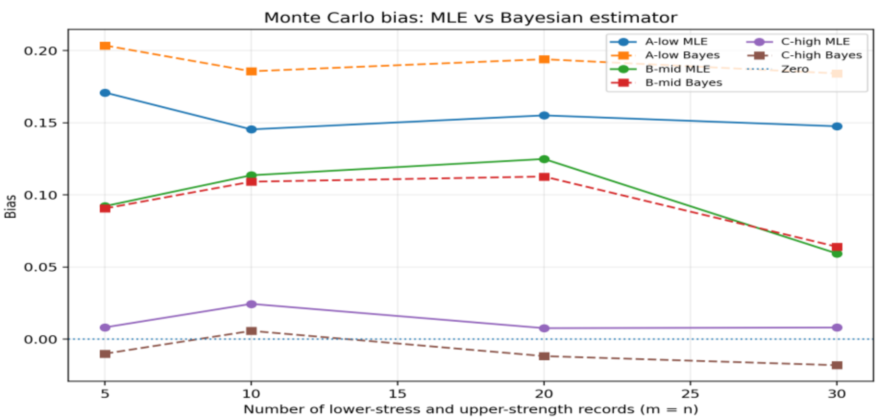

Figure *2*: Monte Carlo bias comparison of the maximum likelihood and Bayesian posterior-mean estimators under different reliability scenarios and record sizes

Figur*e* 3 compares the RMSE of the maximum likelihood and Bayesian posterior-mean estimators for different record sizes in the low, moderate, and high reliability scenarios. The results obtained show that the Bayesian estimator always has lower RMSE values than the MLE in all the cases considered, indicating a better overall estimation accuracy under the mixed-records settings studied. The lowest RMSE values were observed for the highest reliability scenario and the highest RMSE values for the lowest reliability scenario. Although the RMSE does not decrease monotonically with the number of records, at least for some intermediate record sizes, the general trend supports the conclusion that the Bayesian estimator leads to more stable finite-record performance than the MLE in the scenarios considered.

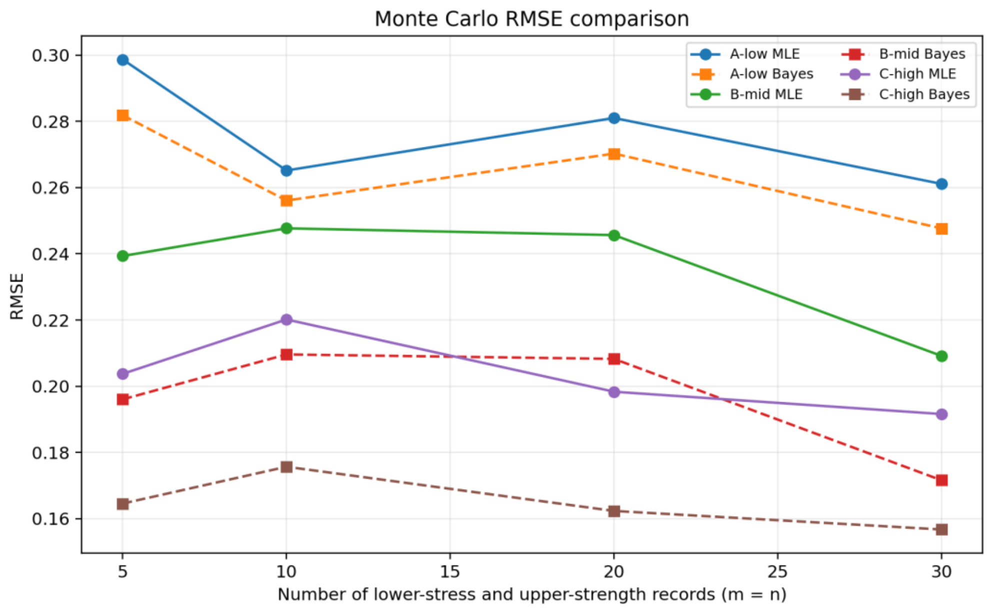

Figure 3*: Monte Carlo RMSE comparison of the maximum likelihood and Bayesian posterior-mean estimators under different reliability scenarios and record sizes.*

Overall, the Bayesian posterior-mean estimator exhibits lower RMSE than the MLE at every simulated design point, indicating superior overall estimation accuracy under the mixed-record settings considered. In the low-reliability scenario, both estimators show positive bias, whereas in the high-reliability scenario, the MLE bias remains close to zero, and the Bayesian estimator exhibits only a small negative bias at some record sizes. The RMSE does not decrease monotonically with increasing record size in all cases. This behavior can be attributed to the nature of record data, since successive record values are highly selected extreme observations and do not contribute information in the same manner as additional independent and identically distributed observations.

5.2 Interval results

The empirical coverage probabilities and the average interval lengths of the nominal 95% asymptotic confidence interval and Bayesian equal-tail credible interval are summarized in **Table3** over the three reliability scenarios and different record sizes. Overall, neither interval procedure works well in terms of nominal 0.95 coverage, especially for a small number of records. The Bayesian interval coverage is generally closer to the nominal level than the asymptotic interval coverage, with the most obvious improvements in the low and high reliability scenarios. For example, for the C-high case with five records, the Bayesian coverage is 0.900 versus 0.800 for the asymptotic interval. In the case of moderate reliability, the coverage is increasing with the size of the record: it is 0.940 for the Bayesian interval and 0.905 for the asymptotic interval at 30 records. In the aspect of the interval width, the Bayesian credible interval is in general shorter than the asymptotic confidence interval except for the B-mid scenario with 10 records, for which the asymptotic confidence interval is shorter. The results show that the Bayesian interval provides a satisfactory compromise between coverage and interval length for most of the finite-record scenarios considered here. The considerable under-coverage of both methods further underlines the fact that inference based on a small number of record values is still a challenging task.

Table 3: Empirical coverage probabilities and average interval lengths of the asymptotic and Bayesian interval procedures.

Scenario	Records	Asymptotic Coverage	Bayes Coverage	Asymptotic Avg. Length	Bayes Avg. Length
A-low	5	0.770	0.800	0.6108	0.5565
A-low	10	0.835	0.875	0.6325	0.5682
A-low	20	0.795	0.855	0.5913	0.5577
A-low	30	0.805	0.870	0.6056	0.5723
B-mid	5	0.845	0.880	0.5770	0.5570
B-mid	10	0.860	0.865	0.5490	0.5497
B-mid	20	0.895	0.860	0.5586	0.5487
B-mid	30	0.905	0.940	0.5810	0.5698
C-high	5	0.800	0.900	0.6009	0.5461
C-high	10	0.840	0.855	0.6021	0.5305
C-high	20	0.850	0.900	0.6235	0.5496
C-high	30	0.860	0.910	0.6301	0.5536

The empirical coverage probabilities of the nominal 95% asymptotic confidence intervals and Bayesian equal-tail credible intervals for the three reliability scenarios and different record sizes are shown in **Figure 4.** In all cases, the coverage probabilities are less than the nominal 0.95 level; neither interval procedure achieves perfect finite-record performance. However, the Bayesian intervals tend to have closer coverage to 0.95 than the asymptotic intervals, especially in the low- and high-reliability settings. The most serious undercoverage is found in the asymptotic interval in the A-low case, where coverage is roughly 0.77–0.84. However, the Bayesian interval demonstrates superior coverage overall, achieving 0.94 with 30 records in the B-mid scenario.

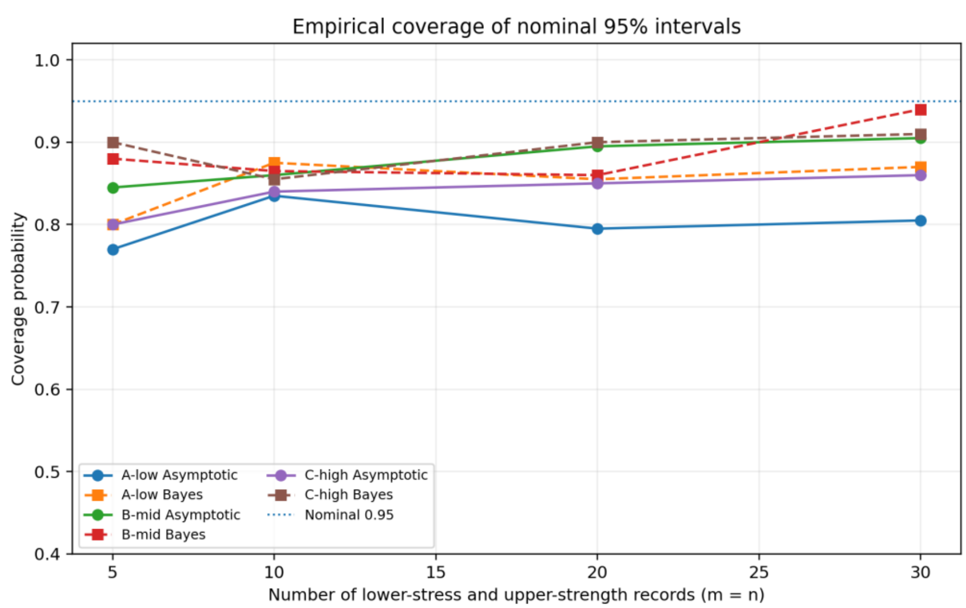

Figure 4: Empirical coverage probabilities of nominal 95% asymptotic and Bayesian intervals.

The average lengths of the nominal 95% asymptotic confidence intervals and Bayesian equal-tail credible intervals are compared in **Figure 5** for the three reliability cases and for various record sizes. Overall, the Bayesian intervals are shorter than the corresponding asymptotic intervals, which means that the precision is higher under most of the settings of finite records studied. This difference is especially noticeable for the high-reliability case, where the Bayesian interval length stays well below the asymptotic interval length for all record sizes. For the moderate-reliability scenario, the two procedures have more comparable interval lengths with only small differences for some record sizes. The average interval length is not monotonically decreasing with the increasing number of records. This reflects the highly selective nature of record observations and the nonuniform amount of information contributed by successive records. In most of the scenarios studied, the Bayesian procedure generally provides narrower intervals with coverage probabilities closer to the nominal level.

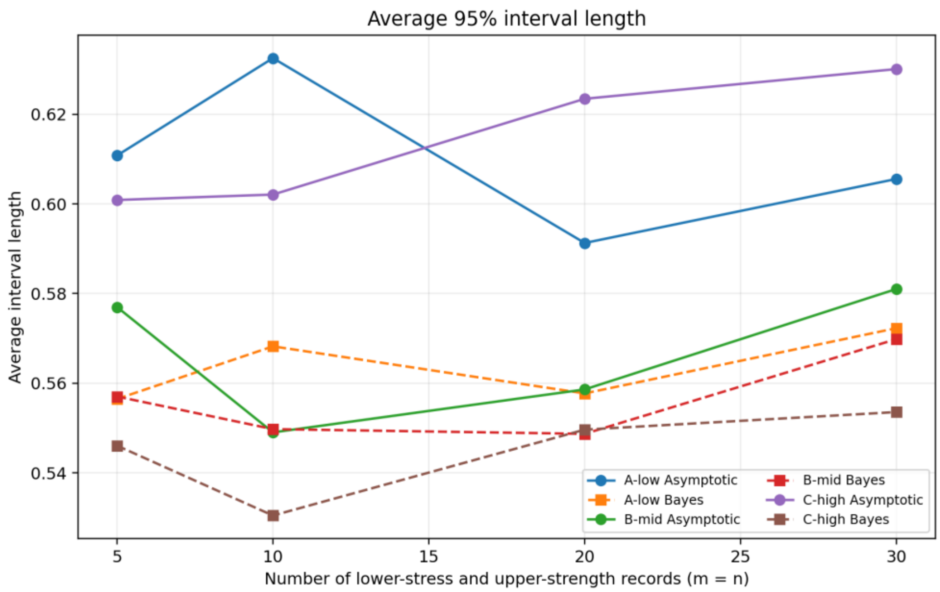

Figure 5: Average interval-length comparison.

Neither interval procedure attains nominal 0.95 coverage uniformly in these finite-record experiments. The Bayesian equal-tail interval generally has coverage closer to 0.95 and is usually shorter. The asymptotic interval undercovers particularly in the low-reliability scenario. This is a key practical finding: ordinary large-sample arguments based on record count should not be accepted without finite-record calibration.

6. DATA AND RECORD EXTRACTION

A real-data example is provided to demonstrate the practical implementation of the proposed mixed-record stress-strength framework based on published lifetime observations of steel specimens. The analysis is intended to demonstrate the suggested Poisson–Exponential (PE) method in an asymmetric record scheme, where the strength process is represented by upper record values and the stress process by lower record values.

6.1 Data and Records Retrieval

The real-data illustration is based on two lifetime sequences of steel specimens by Nayal et al. (2023). The first sequence is 24 lifetime observations at stress level 32, and the second sequence is 20 lifetime observations at stress level 32.5. To demonstrate the asymmetric mixed-record framework studied in this work, we assign the level 32 sequence to the strength process and the level 32.5 sequence to the stress process. This assignment is introduced purely for methodological reasons and should not be interpreted as implying that these were the original physical roles assigned to the two series in the source study. The order of observations affects record statistics, and throughout the analysis we maintain the original reported ordering of both lifetime sequences. The observations are used only as positive continuous lifetime data to demonstrate the proposed PE mixed-record methodology. The full ordered sequences used for parameter calibration and record extraction are presented in **Table 4.**

Table 4: Complete ordered sequences used for calibration and record extraction

Role in this paper	Source series	Ordered observations
Strength X	Stress level 32	1144, 231, 523, 474, 4510, 3107, 815, 6297, 1580, 605, 1786, 206, 1943, 935, 283, 1336, 727, 370, 1056, 413, 619, 2214, 1826, 597
Stress Y	Stress level 32.5	4257, 879, 799, 1388, 271, 308, 2073, 227, 347, 669, 1154, 393, 250, 196, 548, 475, 1705, 2211, 975, 2925

The records retained are:

$$\text{Upper strength records: } 1144,\ 4510,\ 6297,$$

and

$$\text{Lower stress records: } 4257,\ 879,\ 799,\ 271,\ 227,\ 196.$$

The strict upper strength records occur at observation positions 1, 5, and 8, giving (1144, 4510, 6297). The strict lower stress records occur at positions 1, 2, 3, 5, 8, and 14, giving (4257, 879, 799, 271, 227, 196). Thus, the mixed-record analysis retains 3 upper-strength records and 6 lower-stress records from 44 complete observations.

6.2. Parameter Calibration and Model Assessment

The common exponential-rate parameter was calibrated from the two complete sequences by maximizing the joint Poisson-exponential likelihood with separate shape parameters and a shared rate. The resulting complete-sample calibration estimates were theta_X = 0.80785, theta_Y = 0.06966, and beta = 0.00089173. Only beta was carried forward as fixed in the subsequent record-only inference; the record-based shape parameters were re-estimated from the retained upper and lower records, as shown in **Table 5**.

$$\beta = 0.00089174.$$

This separation between calibration and inference is deliberate. The complete sequences are used only to obtain a stable common-rate calibration and a descriptive model-fit check; all subsequent estimates of the stress-strength reliability use the mixed record values. At the joint calibration fit, the Kolmogorov-Smirnov distances are 0.1168 for the strength sequence and 0.1557 for the stress sequence. These distances are reported as descriptive diagnostics only; because the parameters are estimated from the same data, they are not interpreted here as unadjusted goodness-of-fit p-values. Accordingly, the application should be viewed as a methodological real-data illustration rather than evidence that the PE model is uniquely optimal for these lifetimes.

Table 5: Complete-sample PE calibration and record reduction summary.

Process	N	Full-sample theta	Common beta	KS distance	Retained records
Strength X	24	0.80785	0.00089173	0.1168	3 upper
Stress Y	20	0.06966	0.00089173	0.1557	6 lower

All models were fitted to the same 44complete observations, and no location parameter was included.

The maximized log-likelihood, Akaike information criterion (AIC), Bayesian information criterion (BIC), and descriptive Kolmogorov–Smirnov distances for the strength and stress samples are used to compare the fit of the exponential, Weibull, and Poisson–exponential models in **Table 6**. The Weibull model has the largest maximized log-likelihood value (-357.6763), while the common-rate exponential model has the smallest AIC (718.5483) and BIC (720.3324) values, indicating that it provides the best trade-off between fit of the model and complexity among the three models considered. For the strength sample, the Poisson–Exponential model has the smallest KS distance, D_X=0.1168$. However, its stress-sample KS distance, D_Y=0.1557, is a bit larger than those of the competing models. There is no clear empirical advantage of the Poisson-exponential model over the exponential or Weibull alternatives in general. Therefore, the Poisson–exponential distribution is kept in this study as a reasonable working model to demonstrate the proposed mixed-record inferential framework, rather than the uniquely best-fitting distribution for the complete lifetime data.

The common-rate exponential model produced the smallest AIC and BIC, whereas the Weibull and PE models gave only modest improvements in the maximized likelihood. The PE model produced descriptive KS distances of 0.1168and 0.1557for the strength and stress sequences, respectively, which were comparable to those obtained under the competing models. Therefore, the available complete samples do not provide strong evidence that the PE distribution is superior to the exponential or Weibull alternatives. The PE model is retained as a reasonable working distribution for demonstrating the proposed mixed-record methodology, and the real-data application should not be interpreted as evidence that it is the uniquely optimal lifetime model for these observations.

Table 6: Comparison of candidate lifetime models using log-likelihood, AIC, BIC, and descriptive KS distances.

Model	(k)	Maximized log-likelihood	AIC	BIC	(D_X)	(D_Y)
Exponential, common rate	1	(-358.2741)	**718.5483**	**720.3324**	0.1503	0.1436
Weibull, separate shapes and common scale	3	(-357.6763)	721.3525	726.7051	0.1248	0.1507
Poisson–Exponential, separate shapes and common rate	3	(-357.9592)	721.9183	727.2709	0.1168	0.1557

Figure 6 shows the ordered steel-specimen lifetime observations for the strength and stress sequences and the mixed record values chosen for the following analysis. The upward triangles represent the three strict upper records extracted from the strength sequence at the observation positions 1, 5, and 8. The downward triangles represent the six strict lower records extracted from the stress sequence at the positions 1, 2, 3, 5, 8, and 14. This means that the initial 44 observations are condensed into nine record values, emphasizing the loss of information in the analysis based on records. In all cases, the original order of observation is preserved due to the sequence-dependent identification of upper and lower records.

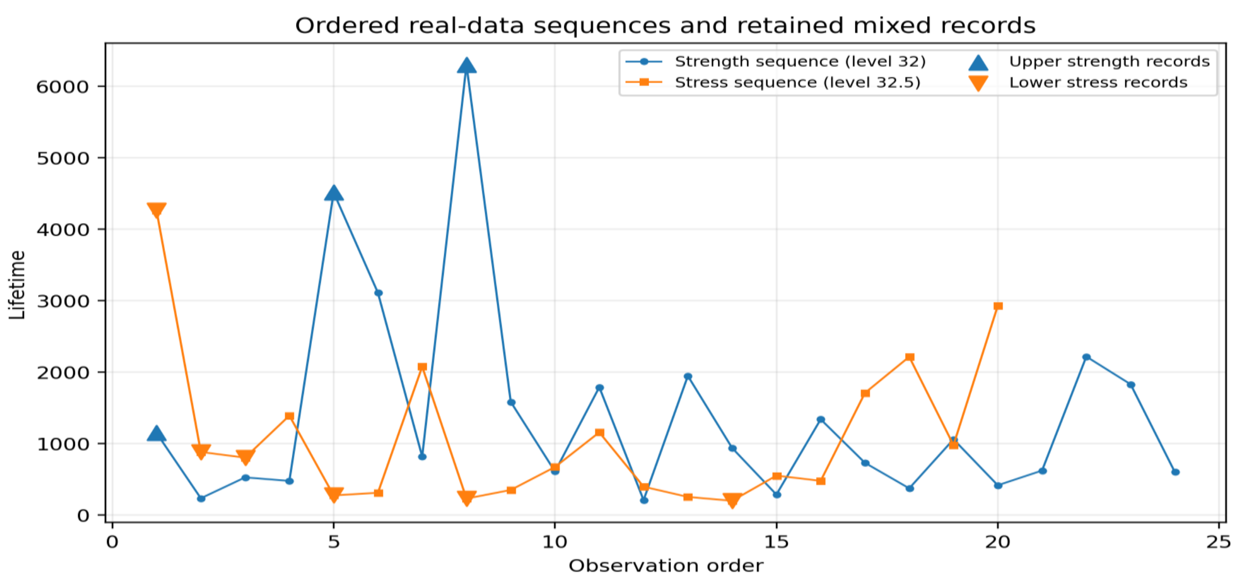

Figure 6: Ordered steel-specimen lifetime sequences and the mixed records retained for analysis. Triangles mark upper strength records and lower stress records in their original reported order.

7. Numerical results

For the record-only analysis, beta = 0.00089173 was treated as fixed. The two shape parameters were estimated by maximum likelihood from the three upper strength records and six lower stress records. Bayesian inference used independent flat working priors for the two positive shape parameters, with posterior integration performed on the log-parameter scale. The reported Bayesian summaries are based on 12,000 posterior draws, and the bootstrap-Bayesian percentile interval is based on 500 parametric bootstrap replications that regenerate mixed record sequences under the fitted model, as shown in **Table 7**.

Table 7: Record-based classical, Bayesian, and bootstrap-Bayesian estimates for the real-data illustration.

Quantity	Estimate/Interval
$\hat{\theta}_X$ (MLE, upper strength records)	3.9342
$\hat{\theta}_Y$ (MLE, lower stress records)	5.2996
$\hat{R}_{ML}$	0.4317
Approximate $SE(\hat{R}_{ML})$	0.2753
95% logit asymptotic CI	(0.0777, 0.8726)
Posterior mean of θ_X	6.7171
Posterior mean of θ_Y	6.3953
Bayes posterior mean $\hat{R}_B$	0.4921
95% Bayesian credible interval	(0.1495, 0.8441)
95% bootstrap-Bayesian percentile CI	(0.2924, 0.9571)

The record-based maximum likelihood estimate is R_hat_ML = 0.4317, while the Bayesian posterior mean is R_hat_B = 0.4921. Under the convention R = P(Y < X), these values correspond to an estimated probability of approximately 43% to 49% that a randomly selected strength exceeds a randomly selected stress. The 95% logit asymptotic interval is (0.0777, 0.8726), the 95% equal-tail Bayesian credible interval is (0.1495, 0.8441), and the 95% bootstrap-Bayesian percentile interval is (0.2924, 0.9571). The considerable width of all three intervals reflects the severe information loss caused by reducing 44 complete observations to only nine record values.

The bootstrap-Bayesian interval is shifted upward relative to the posterior credible interval, indicating skewness in the repeated-sampling distribution of the Bayesian posterior-mean estimator for this short mixed-record configuration. The example therefore illustrates both the feasibility of the proposed inferential procedure and the uncertainty that must be acknowledged when the available information consists only of a few extreme records.

Further caution concerns the observation order. If the order printed in the source table is not the true acquisition order, these extracted records should be viewed as an ordered-data illustration. Sorting the data before extracting records would be invalid.

8. Discussion

More recent studies emphasize the importance of Bayesian and finite-sample inference in stress–strength models with record data. [13] investigated classical, Bayesian, and parametric bootstrap methods for stress-strength reliability based on lower record sampling and demonstrated that Bayesian estimation can provide excellent finite-sample performance. Similarly, [14] studied the inverse Lindley model by using lower record values and combined maximum likelihood, asymptotic, and bootstrap confidence intervals with Bayesian estimation through various numerical techniques. More recently, [15] studied the upper-record stress–strength inference for the Lomax–Rayleigh distribution. They found that Bayesian estimators generally had better mean-squared-error performance and the coverage of exact and bootstrap intervals was more reliable than the coverage of asymptotic intervals for moderate record sizes. Of particular interest to the present work, [16] studied stress-strength reliability for the discrete Alpha-Power Weibull distribution under Type-II censoring, upper records, and mixed upper-lower record data and showed that the relative performance of the classical and Bayesian procedures is strongly dependent on the structure of incomplete data and the sample size. These recent results emphasize the importance of explicit finite-record evaluation and careful uncertainty assessment instead of relying on conventional asymptotic approximations. What is different in the present study is that the mixed lower-stress/upper-strength formulation is developed specifically for the continuous Poisson-exponential model under a common calibrated-rate structure.

9. Conclusions

This study developed a unified classical and Bayesian framework for stress–strength reliability inference under an asymmetric mixed-record sampling scheme for the Poisson–exponential distribution. In the proposed setting, system strength is observed through upper record values, whereas stress is represented by lower record values. The corresponding mixed-record likelihood was derived by combining the two record contributions under independence. Because short record sequences provide limited information for simultaneous estimation of all Poisson–exponential parameters, the principal analysis adopted a commonly known or externally calibrated rate parameter and estimated the two population-specific shape parameters from their respective record samples. Within this framework, maximum likelihood estimation, a logit-based asymptotic confidence interval, Bayesian posterior estimation under independent flat working priors, and a parametric bootstrap-Bayesian percentile interval were developed. The Monte Carlo investigation showed that the Bayesian posterior-mean estimator achieved lower RMSE than the MLE at every reported design point across the low-, moderate-, and high-reliability scenarios. The Bayesian equal-tail intervals also generally produced coverage probabilities closer to the nominal 95% level and shorter average lengths than the corresponding asymptotic intervals. Nevertheless, neither interval procedure attained nominal coverage uniformly, emphasizing the finite-record limitations associated with inference based on highly selected extreme observations. The real-data illustration further demonstrated the practical implications of this information reduction. From 44 published steel-specimen lifetime observations, only three upper-strength records and six lower-stress records were retained. The associated asymptotic, Bayesian, and bootstrap-Bayesian intervals remained wide, reflecting the substantial uncertainty induced by the small number of retained records. In addition, complete-sample model comparisons did not establish the Poisson–exponential distribution as uniquely superior to the exponential and Weibull alternatives; therefore, the real-data analysis should be interpreted as a methodological illustration of the proposed mixed-record framework. Overall, the results demonstrate that asymmetric lower–upper record schemes can be incorporated coherently into stress–strength reliability analysis, while also showing that uncertainty assessment is essential when only a few record observations are available. Future research should consider fully unknown rate structures, the use of inter-record times or other auxiliary information, alternative prior formulations, dependent stress–strength processes, and extensions to multicomponent systems and other lifetime distributions. Such developments may improve identifiability, reduce uncertainty, and broaden the practical applicability of mixed-record reliability inference.

DATA AVAILABILITY

The data used in this study, including the processed record values and simulation outputs, can be provided by the corresponding author upon reasonable request.

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