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Bernoulli Polynomials for Solving Three-Dimensional Volterra-Fredholm Integral Equations of the Second Kind

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ABSTRACT

In this work, our approach for solving three-dimensional linear Volterra-Fredholm integral equations (3D-VFIEs) based on Bernoulli polynomials. This approach was previously applied to solve two-dimensional Volterra-Fredholm integral equations. This method transforms the 3D-VFIEs into a system of linear algebraic equations. The Banach fixed-point theorem is employed to demonstrate the existence and uniqueness of the 3D-VFIE. It has demonstrated its efficiency and effectiveness in achieving accurate numerical results, outperforming other methods. The numerical method that we relied on in solving (3D-VFIEs) as explained in this paper, and the approximate solution that we reached it by comparing it to other solutions that were deduced by other methods showed us very close results for some methods such as the Lucas and Shifted Chebyshev polynomials methods and more efficient than other ones such as Haar Wavelet's technique, Block-pulse functions and Modified block-pulse functions. Compared to other error rates, the error coefficient was the best. This is supported by examples of numerical solutions to linear integral equations, statistical tables, and all figures, which provide the strongest evidence of the convergence of the exact and approximate solutions. The high accuracy of this method is verified through some numerical examples. The Maple 18 program outputs all of the results.

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1. INTRODUCTION

In recent years integral equations (IEs) have a prominent and clear role in many fields of science, some problems associated with IEs appear in physics, chemistry, biology, engineering, economics and also These equations appear in fracture mechanics, aerodynamics, the theory of porous filtering, antenna problems in electromagnetic theory, in the quantum effects of electromagnetic fields in the blackbody whose interior is filled by a Kerr nonlinear crystal, in the description of the three- dimensional structure of water around globular solutes, and the study of a traveling wave solution for a mathematical model describing the population change influenced by a uniformly changing environment [1, 2]. and other sciences, which has allowed scientists and researchers to search for analytical and numerical methods to solve the related problems. Therefore, the creation, improvement, and development of several high-order numerical methods for solving (IEs) have received considerable attention [3, 4].

Numerical methods have been proposed in recent years to solve one- and two-dimensional Volterra and Fredholm integral **Equations** (5-9). Several numerical methods for approximating the solution of linear and nonlinear three-dimensional integral equations, especially three-dimensional Volterra–Fredholm integral **Equations** (5, 6, 10-12). Also, Bernoulli polynomials are studied by many authors and applied to solve different problems [14-16]. In the presented paper, we apply Bernoulli polynomials to solve 3D-VFIEK2 as illustrated in the numerical examples section.

This research paper is structured as follows: In Section 2, we discuss the existence and uniqueness of the solution by using the Banach fixed-point theorem. In Section 3, we present some definitions and properties of Bernoulli numbers and polynomials. Section 4 details the solution of 3D-VFIEs by using the Bernoulli polynomials method. Section 5 provides numerical examples along with illustrative **Figures** 1, 2, 3, 4, 5, and 6.

2. EXISTENCE AND UNIQUENESS OF SOLUTION

Consider the following 3D-IE.

$$\begin{aligned}
 &H(x, y, z) \\
 &= g(x, y, z) \\
 &+ \mu \int_0^1 \int_0^1 \int_0^1 K(x, y, z, u, v, w)H(u, v, w) du dv dw \\
 &+ \mu \int_0^1 \int_0^1 \int_0^1 K_2(x, y, z, u, v, w)H(u, v, w) du dv dw, \tag{1}
 \end{aligned}$$

Where,

$$K(x, y, z, u, v, w) = \begin{cases} K_1(x, y, z, u, v, w), & (u, v, w) \leq (x, y, z) \\ 0, & (u, v, w) > (x, y, z), \end{cases} \tag{2}$$

Where,

$(x, y, z) \in \Omega = [0, 1] \times [0, 1] \times [0, 1]$ and μ is a constant parameter

$H(x, y, z)$ is an unknown function, $g(x, y, z)$, $K_2(x, y, z, u, v, w)$ and $K(x, y, z, u, v, w)$ are analytical functions on $[0, 1] \times [0, 1] \times [0, 1]$.

Equation 1 can be written in the form :

$$H(x, y, z) = g(x, y, z) + \mu \int_0^1 \int_0^1 \int_0^1 L(x, y, z, u, v, w) H(u, v, w) du dv dw \tag{3}$$

$$L(x, y, z, u, v, w) = K(x, y, z, u, v, w) + K_2(x, y, z, u, v, w). \tag{4}$$

To discuss the existence of the unique solution of equation (3), given the Banach fixed-point theorem, we write equation (3) in the integral operator form:

$$\bar{T} H(x, y, z) = g(x, y, z) + \mu T H(x, y, z), \tag{5}$$

Where,

$$T H(x, y, z) = \int_0^1 \int_0^1 \int_0^1 L(x, y, z, u, v, w) H(u, v, w) du dv dw .$$

Then, assume the following conditions:

(i) The kernel satisfies the following inequality

$$|L(x, y, z, u, v, w)| \leq C, (C \text{ is a constant}).$$

(ii)The function $g(x, y, z)$ is continuous in $L_2 ([0, 1] \times [0, 1] \times [0, 1])$. Its norm is $\|g(x, y, z)\|_{L_2} = \{ \int_0^1 \int_0^1 \int_0^1 [g(x, y, z)]^2 dx dy dz \}^{\frac{1}{2}} = G, (G \text{ is constant}).$

(iii) The unknown function $H(x, y, z)$ behaves in $L_2 ([0, 1] \times [0, 1] \times [0, 1])$,

As the free function $g(x, y, z)$ and its norm are defined as $\|G(x, y, z)\| = G.$

The unknown function $H(x, y, z)$ and its norm are defined as $\|H(x, y, z)\| = H.$

Theorem 1: The integral equation (3) has an existence and a unique solution under the condition

$$\mu c < 1. \tag{6}$$

To prove the existence and uniqueness of the solution of the IE (3), we must prove the following two lemmas.

Lemma 1: under the conditions (i) and (iii) in the space $L_2 ([0, 1] \times [0, 1] \times [0, 1])$, the operator \bar{T} maps the space into itself.

Proof: using the conditions (i) and (iii), after applying Hölder-inequality, the integral equation (5) yields:

$$\|\bar{T}H(x, y, z)\| \leq G + \mu c \left\| \left\{ \int_0^1 \int_0^1 \int_0^1 |H(u, v, w)|^2 du dv dw \right\}^{\frac{1}{2}} \right\|$$

$$\|\bar{T}H(x, y, z)\| \leq G + \mu c \|H(x, y, z)\|.$$

Then, in view of the conditions (i), (ii), and (iii), we have

$$\|\bar{T}H(x, y, z)\| \leq G + \delta \|H(x, y, z)\|, (\delta = \mu c). \tag{7}$$

The inequality (7) tells us that the operator \bar{T} maps the ball s_ρ into itself, where

$$\rho = \frac{G}{1 - \mu c}, (\rho > 0). \tag{8}$$

Therefore, we get $\delta < 1$. moreover, the inequality (7) includes the limitation of the integral operator T, where

$$\|TH(x, y, z)\| = \|T\| \|H(x, y, z)\| \leq \mu c \|H(x, y, z)\| \tag{9}$$

Lemma 2: Under the conditions (i), (ii), and (iii), the integral operator \bar{T} It is a contraction in the space $L_2 ([0, 1] \times [0, 1] \times [0, 1])$.

Proof: Assume the two different functions $\{H_1(x, y, z), H_2(x, y, z)\} \in L_2 ([0, 1] \times [0, 1] \times [0, 1])$, Then we have

$$\|\bar{T}H_1(x, y, z) - \bar{T}H_2(x, y, z)\| \leq \mu \left\| \int_0^1 \int_0^1 \int_0^1 |L(x, y, z, u, v, w)| |H_1(u, v, w) - H_2(u, v, w)| du dv dw \right\|.$$

With the aid of conditions (i) and (iii). The above inequality becomes

$$\|\bar{T}H_1(x, y, z) - \bar{T}H_2(x, y, z)\| \leq \mu c \left\| \int_0^1 \int_0^1 \int_0^1 |H_1(u, v, w) - H_2(u, v, w)| du dv dw \right\|.$$

Applying Holder's inequality, we have $\|\bar{T}H_1 - \bar{T}H_2\| \leq \delta \|H_1 - H_2\|$

So we got this equation $\|\bar{T}H_1(x, y, z) - \bar{T}H_2(x, y, z)\| \leq \delta \|H_1 - H_2\|$ (10)

From **Equation 10**, we decide that the operator \bar{T} is continuous in the space $L_2([0, 1] \times [0, 1] \times [0, 1])$, And then it is a contraction operator, under the condition $\delta < 1$. So, from the Banach fixed-point theorem, \bar{T} has a unique fixed-point, which is the unique solution of **Equation 3**.

3. DEFINITION AND PROPERTIES OF BERNOULLI NUMBERS AND POLYNOMIALS

3.1. The Generating Function of the Bernoulli Numbers

Definition. 1: [17-21]. The Bernoulli numbers are the coefficients of the exponential generating function :

Consider the Bernoulli generating function $g(z) = \frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m z^m}{m!}$ where (B_m) is the sequence of the Bernoulli numbers that we factored out in the next step

$$g(z) = b_0 + b_1 \frac{z}{1!} + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \dots = \sum_{m=0}^{\infty} \frac{B_m z^m}{m!},$$

where g is an “exponential” generating function for (B_m) where $m = 0, 1, 2, \dots, \infty$.

We can find the first few terms of the sequence by evaluating the Taylor series expansion of $\frac{z}{e^z - 1}$. We can calculate derivatives for g and their limits as z approaches 0

As follows:

$$\lim_{z \rightarrow 0} g(z) = 1, \quad \lim_{z \rightarrow 0} g^{(1)}(z) = \frac{-1}{2}, \quad \lim_{z \rightarrow 0} g^{(2)}(z) = \frac{1}{6} \quad \text{and so on.}$$

$$g(z) = b_0 + b_1 \frac{z}{1!} + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \dots = \sum_{m=0}^{\infty} \frac{B_m z^m}{m!} = 1 + \left(\frac{-1}{2}\right) \frac{z}{1!} + \left(\frac{1}{6}\right) \frac{z^2}{2!} + \left(\frac{-1}{30}\right) \frac{z^4}{4!} \dots = \sum_{m=1}^{\infty} \frac{g^{(m)} z^m}{m!},$$

where $g^{(m)}$ is the m -derivatives.

By comparing the coefficients on both sides, we find that

$$b_0 = 1, \quad b_1 = \frac{-1}{2}, \quad b_2 = \frac{1}{6}, \quad b_3 = 0, \quad b_4 = \frac{-1}{30}$$

All odd Bernoulli numbers except $b_1 = \frac{-1}{2}$ are zero.

3.2. Bernoulli Polynomials

The Bernoulli polynomials are a generalization of the Bernoulli numbers.

Definition. 2: [22] The Bernoulli polynomials are a sequence of polynomials, $B_m(S)$, defined by the following power series expansion:

$$\frac{z}{e^z - 1} e^{-S z} = \sum_{m=0}^{\infty} \frac{B_m(S) z^m}{m!}. \tag{11}$$

The generating function for the Bernoulli polynomials is the generating function for the Bernoulli numbers multiplied by a term of $e^{-S z}$. Our first observation of the Bernoulli polynomials is that the constant term of $B_m(S)$ is in fact B_m . If we set $S = 0$, then $\frac{z}{e^z - 1} e^{-S z} = \frac{z}{e^z - 1}$ and so $B_m(S) = B_m$. we can use the generating function for the Bernoulli numbers to develop a recurrence relation for the Bernoulli polynomials.

The Bernoulli polynomials, $B_m(S)$ satisfy the recurrence relation:

$$B_m(S) = \sum_{L=0}^m \binom{m}{L} B_L S^{m-L} \tag{12}$$

From **Equation 1**, we have [23].

Using this recurrence, we can calculate the first few Bernoulli polynomials:

$$\begin{aligned} B_0(S) &= 1, \\ B_1(S) &= S - \frac{1}{2}, \\ B_2(S) &= S^2 - S + \frac{1}{6}, \\ B_3(S) &= S^3 - \frac{3}{2}S^2 + \frac{1}{2}S, \\ B_4(S) &= S^4 - 2S^3 + S^2 - \frac{1}{30}. \end{aligned} \tag{13}$$

Properties of the Bernoulli polynomials:

$$\begin{aligned} (1) \quad & B_0(S) = 1, \\ (2) \quad & B'_m(S) = m B_{m-1}(S), \\ (3) \quad & \int_0^1 B_m(S) ds = 0 \end{aligned} \tag{14}$$

The third property is satisfied as follows:

$$(4) \int_0^1 B_m(S) ds = \left[\frac{B_{m+1}(S)}{m+1} \right]_0^1 = \frac{1}{m+1} [B_{m+1}(1) - B_{m+1}(0)] = 0.$$

4. SOLUTION OF LINEAR THREE-DIMENSIONAL INTEGRAL EQUATIONS.

In this section, we are interested in using the Bernoulli polynomials method (BPM) for solving the linear three-dimensional integral **Equation 3**.

The present method aims to get a solution using the method mentioned in the previous section as follows:

$$H(x, y, z) = \sum_{i=0}^N \sum_{j=0}^M \sum_{k=0}^L a_{i,j,k} B_i(x) B_j(y) B_k(z) = \omega(x, y, z) \times A \tag{15}$$

Where,

$a_{i,j,k}$ are the unknown coefficients to be determined, N, M, L are any arbitrary natural number.

Where,

$$i = 0, 1, \dots, N \quad , \quad j = 0, 1, \dots, M \quad , \quad k = 0, 1, \dots, L.$$

{A} matrix of unknown coefficients $a_{i,j,k}$ of order 27×1 .

and $B_i(x)B_j(y)B_k(z)$ are Bernoulli polynomials defined in Equation (12). Also $\omega(x, y, z)$ is $a_1 \times (N + 1)(M + 1)(K + 1)$ matrix introduced as follows:

$$\omega(x, y, z) = [c_{i,j,k}(x, y, z)]$$

where $c_{i,j,k}(x, y, z) = B_i(x) B_j(y) B_k(z)$; $i = 0, 1, \dots, N$, $j = 0, 1, \dots, M$ and $k = 0, 1, \dots, L$.

Substituting equation (15) into equation (3), we obtain: as

$$\sum_{i=0}^N \sum_{j=0}^M \sum_{k=0}^L a_{i,j,k} h_{i,j,k}(x, y, z) = \sum_{i=0}^N \sum_{j=0}^M \sum_{k=0}^L a_{i,j,k} B_i(x) B_j(y) B_k(z) - \mu \int_0^1 \int_0^1 \int_0^1 L(x, y, z, u, v, w) \sum_{i=0}^N \sum_{j=0}^M \sum_{k=0}^L a_{i,j,k} B_i(u) B_j(v) B_k(w) du dv dw \quad (16)$$

μ is a constant parameter we put it = 1

By assembling equation (16) at $N + 1$, $M + 1$ and $k+1$ roots of the Bernoulli polynomials we obtain:

$$\sum_{i=0}^N \sum_{j=0}^M \sum_{k=0}^L a_{i,j,k} h_{i,j,k}(x_n, y_m, z_l) = g(x_n, y_m, z_l)$$

For $n = 0, 1, \dots, N$, $m = 0, 1, \dots, M$, $l = 0, 1, \dots, L$.

We can write the matrix equation in the form of:

$$D_\alpha^{Ts} A = G_f$$

To find the matrix {A} as described previously.

Where $G_f = [g(x_0, y_0, z_0), g(x_1, y_0, z_0), \dots, g(x_N, y_0, z_0); g(x_0, y_1, z_0), g(x_1, y_1, z_0), \dots, g(x_N, y_1, z_0); g(x_0, y_0, z_L), g(x_1, y_0, z_L), \dots, g(x_N, y_0, z_L)]^{Ts}$,

$D_\alpha = h_{i,j,k}(x_n, y_m, z_l)$, $i, n = 0, 1, \dots, N$; $j, m = 0, 1, \dots, M$; $k, l = 0, 1, \dots, L$ And T_s (transpose of the matrix).

Finally, {A} can be found by $A = (D_\alpha^{Ts})^{-1} G_f$

The approximate solution of Equation (15) is given by $H(x, y, z) = \omega(x, y, z)A$

The obtained system of linear algebraic equations contains $(N + 1)^3$ equations in the same number as unknowns and constants. Solving this system, we obtain the value of the constants $a_{i,j,k}$ as shown above.

5. NUMERICAL EXAMPLES

In this section, some numerical examples of linear three-dimensional integral equations are illustrative. To show the efficiency and effectiveness of the presented method, based on three-dimensional Bernoulli polynomials, we introduce the following three examples to illustrate the suggested method and compare the presented method with other methods. We confirmed that the results we achieved were the most efficient, accurate, and closest to the actual results. All examples in this section have been checked by using Maple 18.

Example 1. Assume that the next three-dimensional (FIE) of the second type:

$$H(x, y, z) = x^2 y^2 z^2 - \frac{1}{10000} e^{-(xyz)} + 0.01 \int_0^1 \int_0^1 \int_0^1 e^{-(xyz)} u^2 v w^2 \cdot H(u, v, w) du dv dw$$

$$(x, y, z) \in \Omega = [0, 1] \times [0, 1] \times [0, 1] \quad (17)$$

With the exact solution is $H(x, y, z) = x^2 y^2 z^2$. We obtain the next linear integral equation (LIE) [24].

Implementation of the Bernoulli polynomials technique for equation (17) with

$N=M=L=2$ and taking points from the following formula $(x_i, y_j, z_k) = (\frac{i}{N}, \frac{j}{N}, \frac{k}{N}) : i, j, k = 0.1.2.3 \dots \dots N$.

We got a set of 27 linear algebraic equations with an equivalent number of unknown constants. Solving this system of equations, we obtain the value of the constants as follows:

$$a_{0,0,0} = \frac{1}{27}, a_{0,0,1} = \frac{1}{9}, a_{0,0,2} = \frac{1}{9}, a_{0,1,0} = \frac{1}{9}, a_{0,1,1} = \frac{1}{3}, a_{0,1,2} = \frac{1}{3}, a_{0,2,0} = \frac{1}{9}, a_{0,2,1} = \frac{1}{3}, a_{0,2,2} = \frac{1}{3}, a_{1,0,0} = \frac{1}{9}, a_{1,0,1} = \frac{1}{3}, a_{1,0,2} = \frac{1}{3}, a_{1,1,0} = \frac{1}{3}, a_{1,1,1} = 1, a_{1,1,2} = 1, a_{1,2,0} = \frac{1}{3}, a_{1,2,1} = 1, a_{1,2,2} = 1, a_{2,0,0} = \frac{1}{9}, a_{2,0,1} = \frac{1}{3}, a_{2,0,2} = \frac{1}{3}, a_{2,1,0} = \frac{1}{3}, a_{2,1,1} = 1, a_{2,1,2} = 1, a_{2,2,0} = \frac{1}{3}, a_{2,2,1} = 1, a_{2,2,2} = 1.$$

Substituting from these constants into (6), we obtain the approximate solution of **Equation 17**.

$H(x, y, z) = x^2 y^2 z^2$. That is the same as the exact solution.

Example 2. [12] Consider the following linear three-dimensional Volterra integral equation:

$$H(x, y, z) = x \cos z - \frac{x^3 y^3}{9} \sin z \int_0^x \int_0^y \int_0^z u v^2 H(u, v, w) du dv dw \quad (18)$$

Where,

$$(x, y, z) \in \Omega = [0, 1] \times [0, 1] \times [0, 1].$$

With the exact solution $H(x, y, z) = x \cos z$.

Applying Bernoulli polynomials for equation (18) where $N=M=L=2$

We obtain the approximate solution in the form of:

$$\begin{aligned} \text{Approx.} := & -1.000000000 \cdot 10^{-10} + 0.0012300075845 x^2 y^2 z^2 - 0.001165270738 x^2 y^2 z - \\ & 0.0004103405774 x^2 y z^2 - 0.0004103261553 x y^2 z^2 + 0.0003903796924 x y^2 z + 0.0003904272065 x^2 y z - \\ & 0.000131102416 x y z + 0.000136937758 x y z^2 + 8.001000000 \cdot 10^{-10} x^2 z^2 + 4.000000000 \cdot 10^{-13} y z + 2.000000000 \cdot 10^{-13} y^2 z^2 - \\ & 2.000000000 \cdot 10^{-13} y^2 z - 8.001000000 \cdot 10^{-10} x^2 z - 3.000000000 \cdot 10^{-13} y z^2 + 1.000000000 \cdot 10^{-13} x^2 y^2 - 1.000000000 \cdot 10^{-13} x y^2 \\ & - 1.000000000 \cdot 10^{-13} x^2 y - 0.4297256367 x z^2 + 1.000000000 \cdot 10^{-13} x y - 0.02997205741 x z + 1.000000000 \cdot 10^{-10} z^2 + \\ & 1.000000000 x + 2.000000000 \cdot 10^{-14} y - 1.033200000 \cdot 10^{-10} z. \end{aligned}$$

Table 1: Numerical results of example 2

Absolute error for N=M=L=2 at some particular points.

x	y	z	Bernoulli polynomials method	Shifted Chebyshev polynomials method.
0.1	0.1	0.1	2.2992081×10^{-4}	2.2992069×10^{-4}
0.01	0.1	0.1	2.2994385×10^{-5}	2.2994289×10^{-5}
0.01	0.01	0.1	2.2987489×10^{-5}	2.2987394×10^{-5}
0.01	0.01	0.01	2.927158×10^{-6}	2.927109×10^{-6}
0.001	0.01	0.01	2.928071×10^{-7}	2.927257×10^{-7}
0.001	0.001	0.01	2.927958×10^{-7}	2.927145×10^{-7}
0.001	0.001	0.001	3.00020×10^{-8}	2.99266×10^{-8}

Table 1 illustrates the numerical results for this example and the results of the shifted Chebyshev polynomials method for the same example. We note that the error estimate is very close

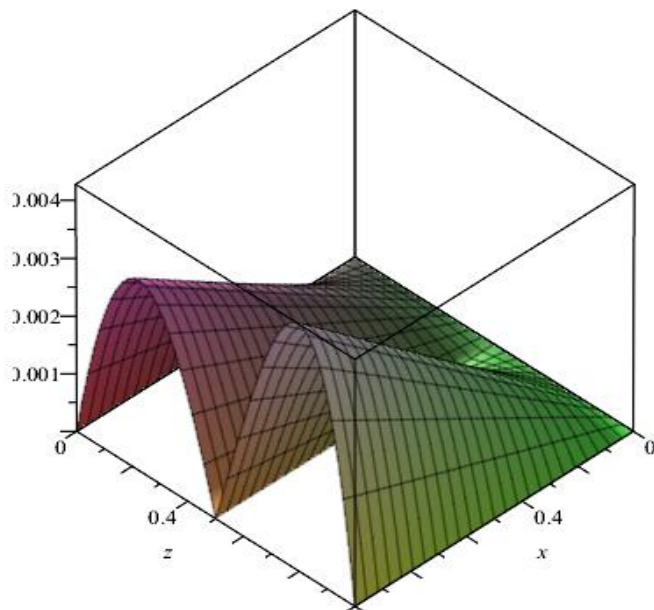


Figure 1. Absolute error of example 2, N=M=L =2 and y =0.5

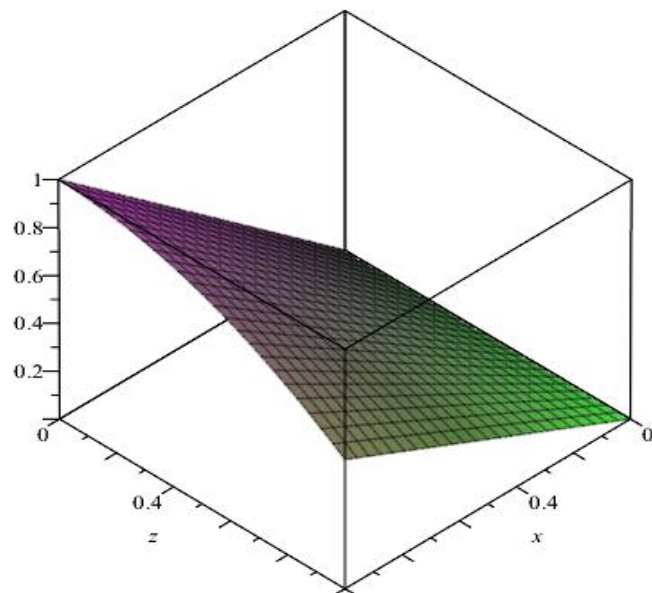


Figure 2. Approximate solution of example 2, $N=M=L=2$, and $y=0.5$

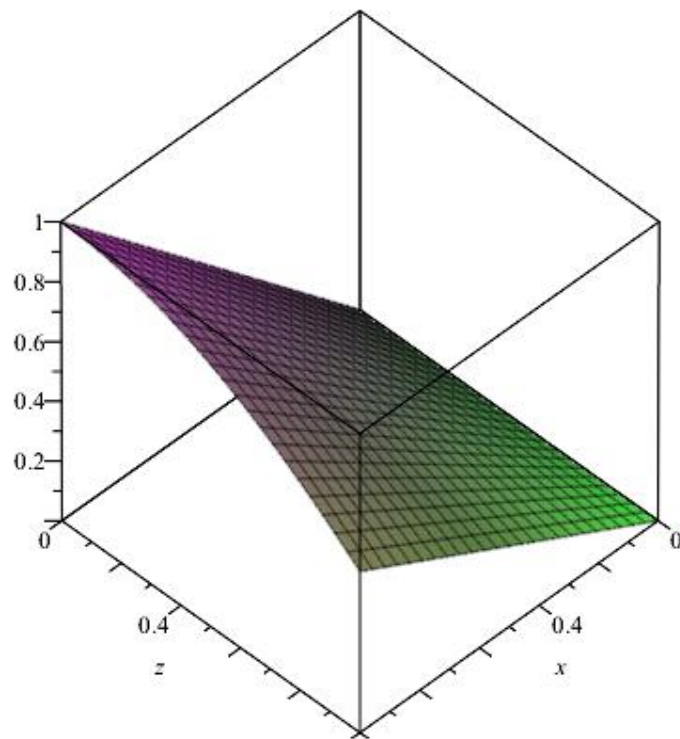


Figure 3. Exact solution of example 2, $N=M=L=2$ and $y=0.5$

Example 3. Consider the following linear three-dimensional Volterra integral equation of the second type:

$$H(x, y, z) = f(x, y, z) + 0.01 \int_0^1 \int_0^1 \int_0^1 e^{-w} x^2 y^2 u v H(u, v, w) \, du \, dv \, dw \quad (19)$$

Where,

$$(x, y, z) \in \Omega = [0, 1] \times [0, 1] \times [0, 1].$$

With the exact solution $H(x, y, z) = x y e^{-z}$.

We obtain the following (LIE) [25]

$$H(x, y, z) = x y e^{-z} - \frac{1}{10000} e^{-(x y z)} + 0.01 \int_0^1 \int_0^1 \int_0^1 (e^{-w} x^2 y^2 u v) (u v e^{-w}) \, du \, dv \, dw.$$

Applying Bernoulli polynomials for equation (19) where $N=M=L=2$.

We obtain the approximate solution in the form of:

$$H(x, y, z) = -0.9417568022 x y z + 0.3096362434 x y z^2 + 0.9999999998 x y + 7.40044 \times 10^{-7} x^2 y^2$$

Table 2 illustrates the numerical results for Example 3 and the results of the Lucas polynomials method for the same example. We note that the error estimate is very close.

Table 2: Numerical results of example 3

Absolute error for $N=M=L=2$ at some particular points.

x	y	z	Bernoulli polynomials method	Lucas Polynomials method[25]
2^{-1}	2^{-1}	2^{-1}	4.63×10^{-8}	4.641×10^{-8}
2^{-2}	2^{-2}	2^{-2}	3.1952049×10^{-4}	$3.19520440 \times 10^{-4}$
2^{-3}	2^{-3}	2^{-3}	7.221211×10^{-5}	$7.221208467 \times 10^{-5}$
2^{-4}	2^{-4}	2^{-4}	1.1471313×10^{-5}	$1.147130303 \times 10^{-5}$
2^{-5}	2^{-5}	2^{-5}	$1.60082511 \times 10^{-6}$	$1.600822047 \times 10^{-6}$
2^{-6}	2^{-6}	2^{-6}	2.109883×10^{-7}	$2.109874999 \times 10^{-7}$

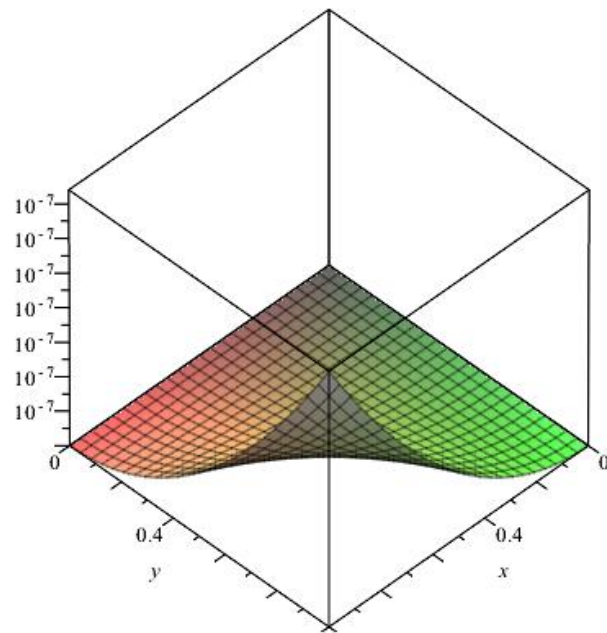


Figure 4. Absolute error of example 3, $N=M=L=2$ and $z=0.5$

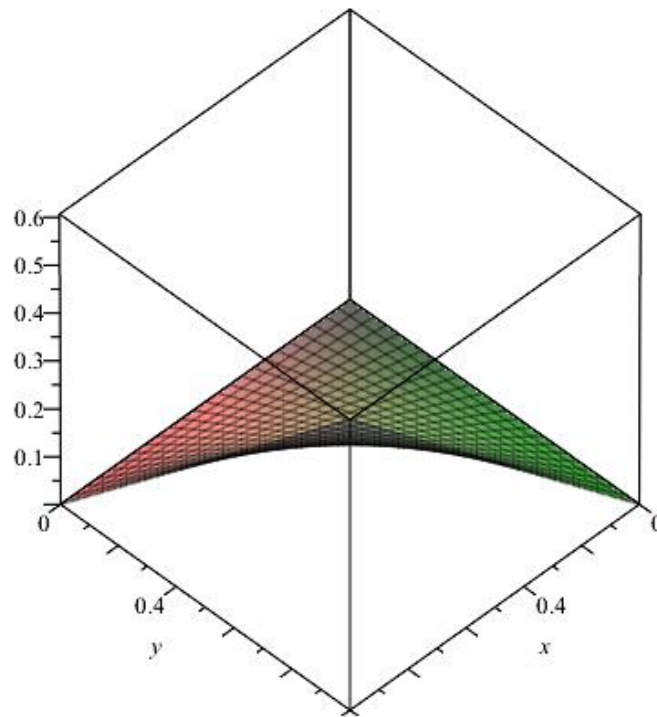


Figure 5. Approximate solution of example 3, $N=M=L=2$ and $z=0.5$

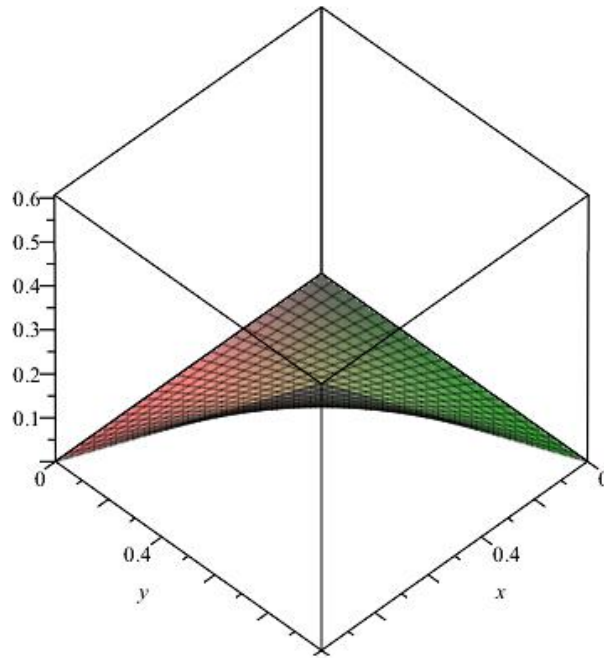


Figure 6. Exact solution of example 3, N=M=L=2 and z=0.5

Example 4. Consider the following three-dimensional Volterra integral equation of the second kind:

$$H(x, y, z) = 4 x^5 y^3 z + 4 x^3 y^3 z^3 + 3 x^4 y^3 z^2 + x^2 y + y z^2 + x y z - 24 \int_0^x \int_0^y \int_0^z x^2 y H(s, t, r) dr dt ds.$$

Where $(x, y, z) \in \Omega = [0, 1) \times [0, 1) \times [0, 1)$.

the exact solution is $H(x, y, z) = x^2 y + y z^2 + x y z$.

Table 3 illustrates the numerical results for this example.

Table 3: Numerical results for Example 4 with 3D-BPM

Nodes (x,y,z) ⁻ⁱ x = y = z = 2	Error for m = 2	Error for m = 4	Error for m = 4
	Present method	3D-BPFs [3]	M3D-BPFs [4]
i = 1	0.04194805589	0.3768138	0.1918396
i = 2	0.004826805848	0.1153397	0.0589723
i = 3	0.0097480101471	0.0013020	0.0224612
i = 4	0.001895593630	0.0064290	0.0032959
i = 5	0.0002871623854	0.0070699	0.0039367
i = 6	0.00003929022628	0.0071500	0.0040168

6.DISCUSSION

In example (1), we find that the approximate solution is the same as the exact solution.

In example(2), at the next point, we observe the following:

x	y	z	Bernoulli polynomials Method.	Shifted Chebyshev polynomials method.
0.1	0.1	0.1	2.2992081×10^{-4}	2.2992069×10^{-4}

The numerical results for this example and the results of the shifted Chebyshev polynomials method for the same example. We note that the error estimate is very close.

x	y	z	Bernoulli polynomials method	Lucas Polynomials method.
0.5	0.5	0.5	4.63×10^{-8}	4063×10^{-8}

In example (3), at the next point, we observe that the numerical results for this example are the same as the results of the Lucas polynomials method for the same example. We note that the error estimate is very close.

In example (4) at the next point, we observe that our method is more efficient than other ones, such as Haar Wavelet’s technique, Block-pulse functions, and Modified block-pulse functions. We applied four numerical examples to solve three-dimensional equations using the Bernoulli polynomials method, where the following was observed: our error estimate is better than the Block-pulse functions and Modified block-pulse functions. These results show the efficiency and the performance of the proposed method.

(x , y , z)	Error for m = 2 Present method	Error for m = 4 3D-BPFs [3]	Error for m = 4 M3D-BPFs [4]
$x = y = z = 2^{-6}$	0.00003929022628	0.0071500	0.0040168

7. CONCLUSION

The numerical method that we relied on in solving (3D-MVFIEK2), as explained in this paper and the approximate solution that we reached by comparing to other solutions that were deduced by other methods showed us very close results for some methods, such as the Lucas and Shifted Chebyshev polynomials methods and more efficient than other ones, such as Haar Wavelet's technique, Block-pulse functions and Modified block-pulse functions. We applied four numerical examples to solve three-dimensional equations using the Bernoulli polynomials method, where the following was observed: our error estimate is better than that of Block-pulse functions and the Modified block-pulse functions. These results show the efficiency and the performance of the proposed method. The proposed method can be extended to systems of nonlinear Volterra integral equations. Also, the development of the method can solve the three-dimensional nonlinear Volterra-Fredholm integral equations. Finally, using the Maple 18 program to solve the examples mentioned in the paper was a great choice, as it was less time-consuming than hand solving.

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