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# Hybrid Entropy–Decimal Approaches for Isomorphism Detection of Factorial Designs

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## ABSTRACT

In experimental design, selecting an optimal design from a vast space of candidates is a fundamental challenge. Many candidate designs are isomorphic-structurally equivalent under relabeling factors, reordering runs, or switching levels and thus statistically indistinguishable. Retaining such redundant designs wastes computational resources and obscures truly distinct experimental strategies. Existing criteria for detecting non-isomorphic designs fail to distinguish all non-isomorphic classes (e.g., the 68 classes of 27-run three-level designs). More powerful projection-based methods achieve perfect discrimination but are computationally expensive. This paper bridges the gap by introducing simple yet powerful hybrid methods that combine entropy with a decimal-system transformation. The approach converts the structural information of a design into a compact decimal vector and then computes an enhanced entropy that incorporates the magnitude of the values, producing a scalar signature invariant under isomorphism. The proposed algorithms work seamlessly for both regular and irregular factorial designs. Extensive experiments on three challenging families-18-run three-level designs (3 non-isomorphic classes), 27-run three-level designs (68 non-isomorphic classes), and 32-run four-level designs (20 non-isomorphic classes) show that the method achieves perfect discrimination at low computational cost. The framework is simple, universal, and efficient, making it a valuable tool for design theorists and practitioners.

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## 1 Introduction

In experimental design [1], a full factorial design (FuFD) provides the most comprehensive framework for studying the effects of multiple input factors on one or more response variables [2]. By evaluating all possible combinations of factor levels, FuFDs allow for the estimation of all main effects and interactions without confounding [3]. However, as the number of factors increases, the total number of experimental runs grows exponentially [4]. For instance, for  $s$  factors each at  $q$  levels, the design requires  $q^s$  runs—quickly becoming impractical in terms of cost, time, and resources [5, 6]. To address this scalability issue, fractional factorial designs (FrFDs) are widely employed [7, 8]. FrFDs enable experimenters to explore the most influential factors using only a carefully chosen subset (fraction) of the full factorial runs [9, 10]. This reduction significantly decreases experimental cost while still capturing essential information about main effects and low-order interactions, which are often of primary interest in screening studies [11, 12].

FrFDs are generally categorized into regular and non-regular types [13]. Regular FrFDs are constructed based on defining relations generated from a linear combination of factor levels in an arithmetic framework. They feature a highly structured aliasing system in which factorial effects (e.g., main effects and interactions) are either fully aliased or completely un-confounded [14, 15]. This clarity in the alias structure makes regular designs straightforward to construct, analyze and interpret using classical factorial design theory, and they are widely used in classical industrial experiments [16, 17]. In contrast, non-regular FrFDs exhibit more complex aliasing structures [18]. These designs may involve partial aliasing among effects [19, 20], which allows for more flexible run sizes and a richer exploration of the factor space, especially when the number of factors is large relative to the experimental budget [21]. Although non-regular FrFDs often require more advanced methods for analysis and interpretation, they can offer superior performance in many practical settings [22, 23].

Selecting an appropriate regular or non-regular FrFD involves navigating a vast design space, especially as the number of factors and allowable run sizes increase [24]. The number of possible designs that are optimal from a given perspective can be extremely large—even for moderate-sized problems—posing significant computational and analytical challenges [25]. To reduce the search space and avoid redundant evaluations, it is essential to recognize and eliminate isomorphic designs, which are designs that are structurally equivalent under a set of transformations such as relabeling factors, permuting runs, or switching factor levels [26]. Isomorphic designs yield identical statistical properties from some perspectives, such as ANOVA, and thus are considered interchangeable from an analytical perspective [27]. In contrast, non-isomorphic designs are genuinely distinct; they differ in their structural and statistical properties, potentially offering different modeling capabilities and interpretability [28]. Identifying and focusing only on non-isomorphic designs enables researchers to explore the true diversity of available design options without redundancy [29]. Consequently, the detection and classification of isomorphic designs play a crucial role in the design construction process, particularly when using computer algorithms to search for optimal designs under various criteria. Efficient algorithms for isomorphism detection not only improve computational efficiency but also ensure that the selected designs represent genuinely distinct experimental strategies.

There are various criteria that are used for constructing optimal designs from optimality various perspective in design of experiment theory, such as distance (maximin, minimax, Hamming, enumerator and other) for space-filling designs [30, 31, 32, 33], (generalized) word-length pattern (GWL) for (generalized) minimum aberration designs [34, 35, 36, 37], and uniformity measures (called discrepancies) for constructing uniform designs [38, 39, 40, 41]. These criteria have recently been successfully used for detecting non-isomorphic regular and non-regular FrFDs [26, 42, 43, 45, 46, 47, 48]. However, existing methods have certain limitations. Some cannot be applied to all types of designs, and some fail to distinguish all non-isomorphic designs. For instance, it was proved in [49] that there are 68 non-isomorphic classes of symmetric orthogonal designs involving 13 factors with 3

levels and 27 runs (denoted  $\mathcal{L}_{27}(3^{13})$ ). These 68 classes have identical GWLP and Hamming distance pattern (HDP) values. For this reason, the distribution of projection GWLP (PGWLP) and projection HDP (PHDP) was explored in [50] to detect non-isomorphism among the 68 classes of  $\mathcal{L}_{27}(3^{13})$ . It turned out that 67 out of 68 combinatorially non-isomorphic classes can be distinguished by either PGWLP or PHDP. Through the average squared correlation (ASC) [51] and the entropy based on PHDP, 67 out of 68 classes can be detected. Hence the projection ASC distribution (PASCSD) was adopted in [46] as an amendment to improve effectiveness. Even though projection of some measures, such as PASCSD, can detect all non-isomorphic designs, it is a computationally expensive technique. For instance, for the class  $\mathcal{L}_{27}(3^{13})$ , there are  $\sum_{k=1}^{13} \binom{13}{k} (68)(6)^k$  projected level-permuted designs.

In this paper, we propose new simple methods that integrate entropy and the decimal system to detect non-isomorphism for regular and non-regular FrFDs. The new methods are able to detect all three classes of  $\mathcal{L}_{18}(3^7)$  [52], all 20 classes of  $\mathcal{L}_{32}(4^9)$  [53], and all 68 classes of  $\mathcal{L}_{27}(3^{13})$  [49], with competitive computational consumption.

The remainder of this paper is organized as follows. Section 2 introduces the necessary notations, definitions, and criteria used throughout the paper. Section 3 presents the proposed algorithms for regular designs, including the decimal system transformation, enhanced entropy, and three hybrid algorithms, along with illustrative examples. Section 4 extends the methodology to non-regular designs using Hamming distance transformations and demonstrates its performance on several design families. Section 5 concludes the paper and discusses directions for future work.

## 2 Notations, Definitions, and Criteria

In this section we introduce the basic notations and concepts that are used throughout the paper.

A design for an experiment with  $n$  runs (experimental trials) and  $m$  factors (inputs), each factor taking  $s$  levels (discrete codes), is denoted as  $D_n(s^m)$ . The  $s$  levels are labelled (coded) symmetrically from the set  $\mathbb{Z}_s = \{0, 1, \dots, s - 1\}$ . This design is represented as an  $n \times m$  matrix  $\mathbf{d} = (d_{ik})_{n \times m}$  with  $d_{ik} \in \mathbb{Z}_s$ .

For a regular fractional factorial design (FrFD), the defining relation, often represented by a set of equations, describes how the factors are related in terms of their interactions. This relation dictates the aliasing structure and determines which main effects and interactions are confounded, ensuring that the design is efficient. In contrast, nonregular FrFDs do not possess a defining relation in the same algebraic sense; they may exhibit partial aliasing and are often constructed using orthogonal arrays or other combinatorial structures.

A critical aspect of design analysis is the word length pattern (WLP). For a regular design, each word in the defining relation has a length equal to the number of factors involved. The WLP is a vector  $(A_1, A_2, \dots, A_m)$ , where  $A_j$  denotes the number of words of length  $j$ . The resolution of a design is the smallest  $j$  such that  $A_j > 0$ . Higher resolution indicates less severe confounding among lower-order effects.

The Hamming distance matrix (HDM) of a design is a key tool for measuring the dissimilarity between any two runs. For two runs  $\mathbf{d}_i$  and  $\mathbf{d}_j$ , the Hamming distance is defined as

$$d_H(i, j) = \sum_{k=1}^m \mathbb{I}(d_{ik} \neq d_{jk}),$$

where  $\mathbb{I}(\cdot)$  is the indicator function (equal to 1 if the condition holds and 0 otherwise). This distance counts the number of factor level changes between the two runs. Designs with large Hamming distances between runs are considered more efficient because they reduce potential confounding and provide better coverage of the experimental region.

For a regular FrFD, the letter pattern matrix (LPM) is a square binary matrix  $L = (a_{ij})$ , where  $a_{ij}$  denotes the number of words that involve factor  $i$  and have length  $j$ . That is,

$$a_{ij} = \begin{cases} 1, & \text{if factor } i \text{ appears in a word of length } j, \\ 0, & \text{otherwise.} \end{cases}$$

The LPM provides more detailed information than the WLP alone and has been used for isomorphism detection, though it is not always sufficient.

Orthogonal designs (OD) are a class of experimental designs that ensure minimal correlation between factors, allowing independent estimation of main effects and interactions. Orthogonality is a key concept in both regular and non-regular FrFDs, as it ensures that the effects of different factors can be separated, leading to clear interpretation and reliable estimation. An orthogonal array (OA) of strength  $t$  is defined as a design  $D_n(s^m)$  such that in any  $t$  distinct columns, each possible  $t$ -tuple of levels appears equally often. If  $D_n(s^m)$  is an OA of strength two, it is called a combinatorially orthogonal design and denoted  $L_n(s^m)$ . The set of all such designs is denoted  $\mathcal{L}_n(s^m)$ . If the design is an OA of strength one, each level appears  $\frac{n}{s}$  times in each column; such designs are referred to as balanced-level (or U-type) designs.

Two designs are said to be combinatorially isomorphic if one can be obtained from the other by a combination of relabeling factors, reordering runs, and switching factor levels. Isomorphic designs share identical statistical properties and are therefore redundant for the purpose of design selection. Non-isomorphic designs are genuinely distinct; they differ in their structural and statistical properties, potentially offering different modeling capabilities and interpretability.

The detection of isomorphism is essential for reducing the search space when constructing optimal designs. In the following sections we develop algorithms that produce a numerical signature (a decimal vector and an associated enhanced entropy) that is invariant under isomorphism and uniquely identifies each non-isomorphic design class.

### 3 The Proposed Methods for Detecting Non-Isomorphic Regular FrFDs

For a binary number with  $k$  digits  $b_{k-1} \dots b_3 b_2 b_1 b_0$ , the decimal number is obtained by summing each binary digit multiplied by its power of two:

$$DS = b_0 \cdot 2^0 + b_1 \cdot 2^1 + b_2 \cdot 2^2 + \dots + b_{k-1} \cdot 2^{k-1}.$$

This transformation converts a binary vector into a unique integer, which preserves the information content while reducing dimensionality. In the context of experimental design, we apply this transformation to binary matrices derived from the defining relations or from Hamming distances, thereby obtaining a compact numerical signature for each design.

Entropy is a classical measure of uncertainty or information content. For a discrete random variable taking values  $X_1, X_2, \dots, X_n$  with probabilities  $p_1, p_2, \dots, p_n$ , the Shannon entropy is defined as

$$H = - \sum_{i=1}^n p_i \log p_i.$$

However, this classical entropy depends only on the probability distribution and ignores the actual magnitudes of the values. In our application, the decimal numbers obtained from a design carry important structural information; two different distributions with the same probabilities but different values would yield the same classical entropy, potentially failing to distinguish non-isomorphic designs.

To overcome this limitation, we propose an *enhanced entropy* that incorporates the values themselves:

$$H_{\text{enh}} = - \sum_{i=1}^n p_i \log p_i \cdot X_i^2.$$

The square weighting  $X_i^2$  amplifies the contribution of larger decimal values, which often correspond to more complex factor–word relationships. This modification makes the entropy sensitive not only to the frequency distribution but also to the actual numeric codes, thereby improving discrimination power. As will be demonstrated in the examples,  $H_{\text{enh}}$  yields distinct values for non-isomorphic designs while remaining invariant under isomorphism.

### 3.1 The Proposed Three Hybrid Methods

Regular fractional factorial designs possess a defining relation, which can be represented as a set of words. We exploit this algebraic structure to construct binary matrices, then convert them to decimal vectors, and finally compute the enhanced entropy. Three complementary methods are developed, each capturing different aspects of the design’s aliasing structure. Their workflows are illustrated in Figures 1, 2 and 3.

**Method 1: EE–DSP–BFWM** This algorithm directly uses the factor–word incidence matrix.

1. Given a regular FrFD, find its defining relation  $I(\mathbf{d})$  and list all words.
2. Sort the words in ascending order (e.g., lexicographically by their letters).
3. Construct a binary factor–word matrix (BFWM) of size  $m \times r$ , where  $m$  is the number of factors and  $r$  the number of words. Set  $w_{ij} = 1$  if factor  $i$  appears in word  $j$ , and 0 otherwise.
4. Convert each binary row of the BFWM to a decimal number; the resulting vector is called the decimal system pattern (DSP).
5. Compute the enhanced entropy  $H_{\text{enh}}$  of the DSP.

The BFWM captures which factors are involved in each word; isomorphic designs have identical BFWM up to row and column permutations, and the subsequent decimal conversion together with the entropy yields an invariant signature.

**Method 2: EE–LWLMP–LWLM** This algorithm summarizes information by word length.

1. Find the defining relation and list all words. Let  $L_{\text{max}}$  be the maximum word length.
2. Construct a letter–word length matrix (LWLM) of size  $m \times L_{\text{max}}$ , where entry  $l_{ij}$  is the number of words of length  $j$  that contain factor  $i$ .
3. Flatten the matrix row-wise into a vector (LWLMP).
4. Compute the enhanced entropy of this vector.

The LWLM aggregates words by length, providing a coarser but still highly discriminative summary. It is especially useful when the number of words is large.

**Method 3: EE–DSP–BFWLM** This algorithm combines the binary nature of Method 1 with the word-length aggregation of Method 2.

1. From the defining relation, generate a binary factor–word length matrix (BFWLM) of size  $m \times L_{\max}$ , where  $b_{ij} = 1$  if factor  $i$  appears in any word of length  $j$ , and 0 otherwise.
2. Convert each binary row to a decimal number (DSP).
3. Compute the enhanced entropy of the DSP.

The BFWLM is more compact than the BFWM (only  $L_{\max}$  columns instead of  $r$ ), while still preserving essential information about the distribution of factor appearances across word lengths.

All three algorithms produce a scalar summary (the enhanced entropy) that is invariant under factor relabeling, run permutation, and level switching. Therefore, two isomorphic designs always yield the same  $H_{\text{enh}}$ , while non-isomorphic designs generally produce different values, as will be verified in the examples below.

### 3.2 The Performance of the Proposed Methods

We now illustrate the three algorithms using three regular  $2^{7-3}$  fractional factorial designs defined by the following generators:

$$\mathbf{d}_1 : X_5 = X_1 X_2, X_6 = X_1 X_2 X_3, X_7 = X_1 X_2 X_4,$$

$$\mathbf{d}_2 : X_5 = X_1 X_2, X_6 = X_1 X_3, X_7 = X_1 X_4,$$

$$\mathbf{d}_3 : X_5 = X_1 X_2, X_6 = X_1 X_2 X_3, X_7 = X_1 X_2 X_4.$$

Designs  $\mathbf{d}_1$  and  $\mathbf{d}_2$  are isomorphic (exchanging  $X_1$  and  $X_5$ ), while  $\mathbf{d}_3$  is non-isomorphic to them. Their defining relations are:

$$I(\mathbf{d}_1) = 125 = 356 = 457 = 1236 = 1247 = 3467 = 1234567,$$

$$I(\mathbf{d}_2) = 125 = 136 = 147 = 2356 = 2457 = 3467 = 1234567,$$

$$I(\mathbf{d}_3) = 125 = 256 = 457 = 1236 = 1247 = 2467 = 1234567.$$

Figure 4 shows the factor–word incidence maps for the three designs.

**Example 1 (Method 1).** Table 1 presents the binary factor–word matrices (BFWM), the corresponding decimal system patterns (DSP), and the enhanced entropy  $H_{\text{enh}}$  for each design. The two isomorphic designs  $\mathbf{d}_1$  and  $\mathbf{d}_2$  yield identical BFWM, DSP, and  $H_{\text{enh}}$  values (6522.9615), while  $\mathbf{d}_3$  gives a different value (9348.7452). Thus Method 1 successfully distinguishes the non-isomorphic design.

**Example 2 (Method 2).** Table 2 gives the letter–word length matrices (LWLM) and the resulting enhanced entropies. Again,  $\mathbf{d}_1$  and  $\mathbf{d}_2$  share the same  $H_{\text{enh}} = 2.0955$ , whereas  $\mathbf{d}_3$  yields 2.4512. Hence Method 2 is also effective.

**Example 3 (Method 3).** Table 3 shows the binary factor–word length matrices (BFWLM), the corresponding DSPs, and the enhanced entropies. The isomorphic pair  $\mathbf{d}_1$  and  $\mathbf{d}_2$  both give  $H_{\text{enh}} = 162.919$ , while  $\mathbf{d}_3$  gives 253.066. Once again, the method clearly separates non-isomorphic designs.

All three methods are simple to implement and computationally efficient. They provide a unified framework for regular designs, and the same principles will be extended to non-regular designs in the following sections.

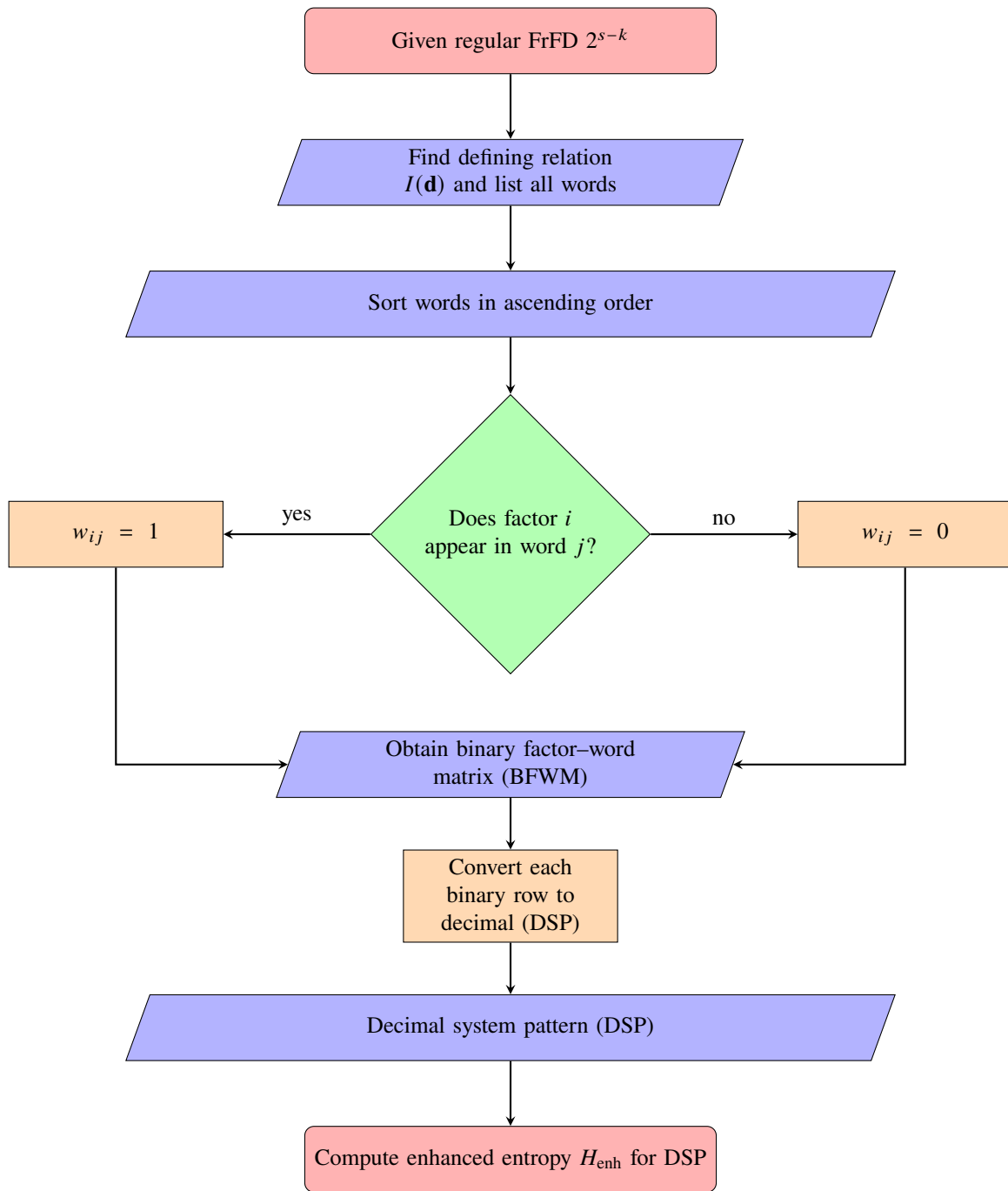


Figure 1: Flowchart for Method 1 (EE–DSP–BFWM) for regular FrFDs.

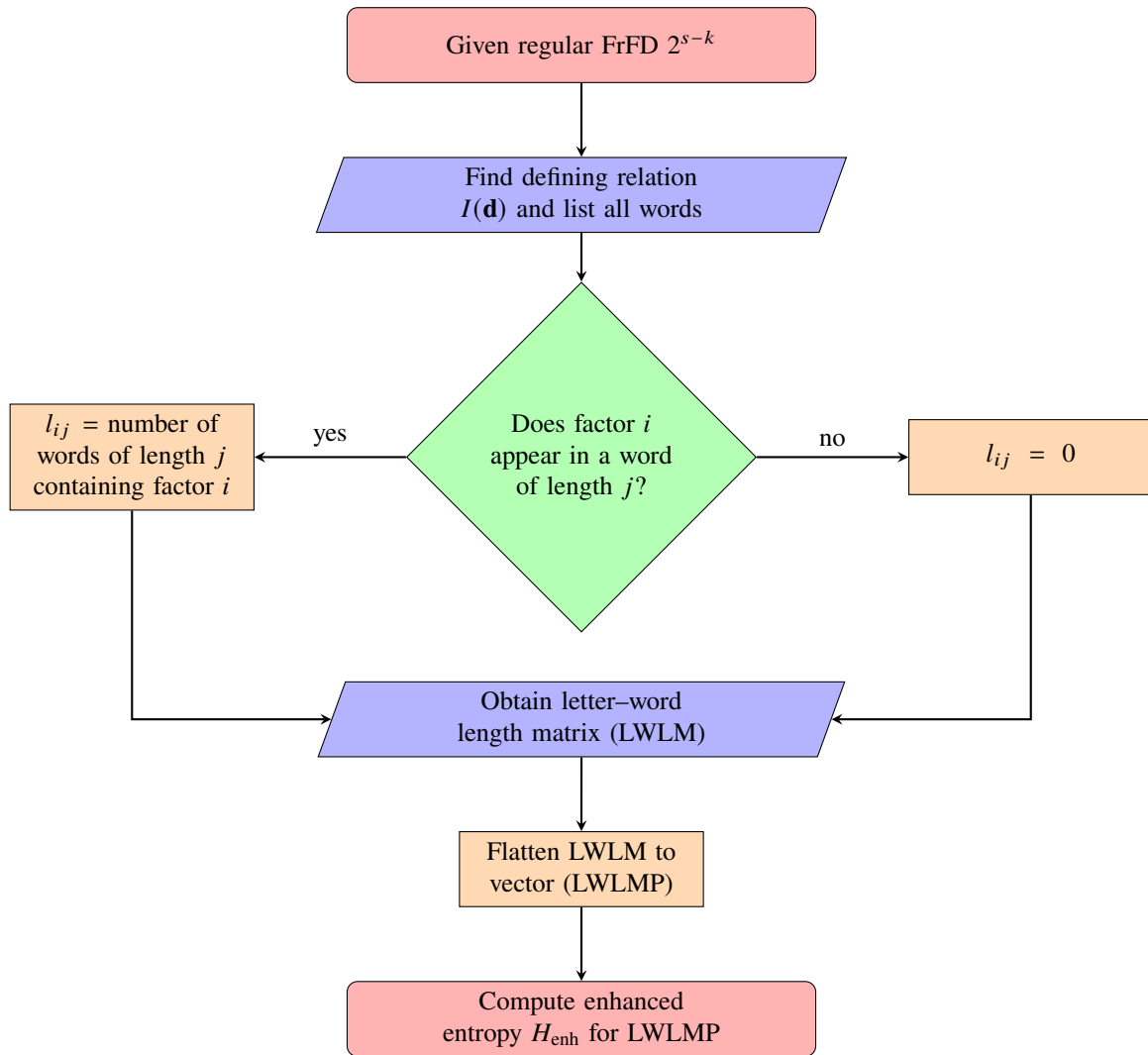


Figure 2: Flowchart for Method 2 (EE-LWLMP-LWLM) for regular FrFDs.

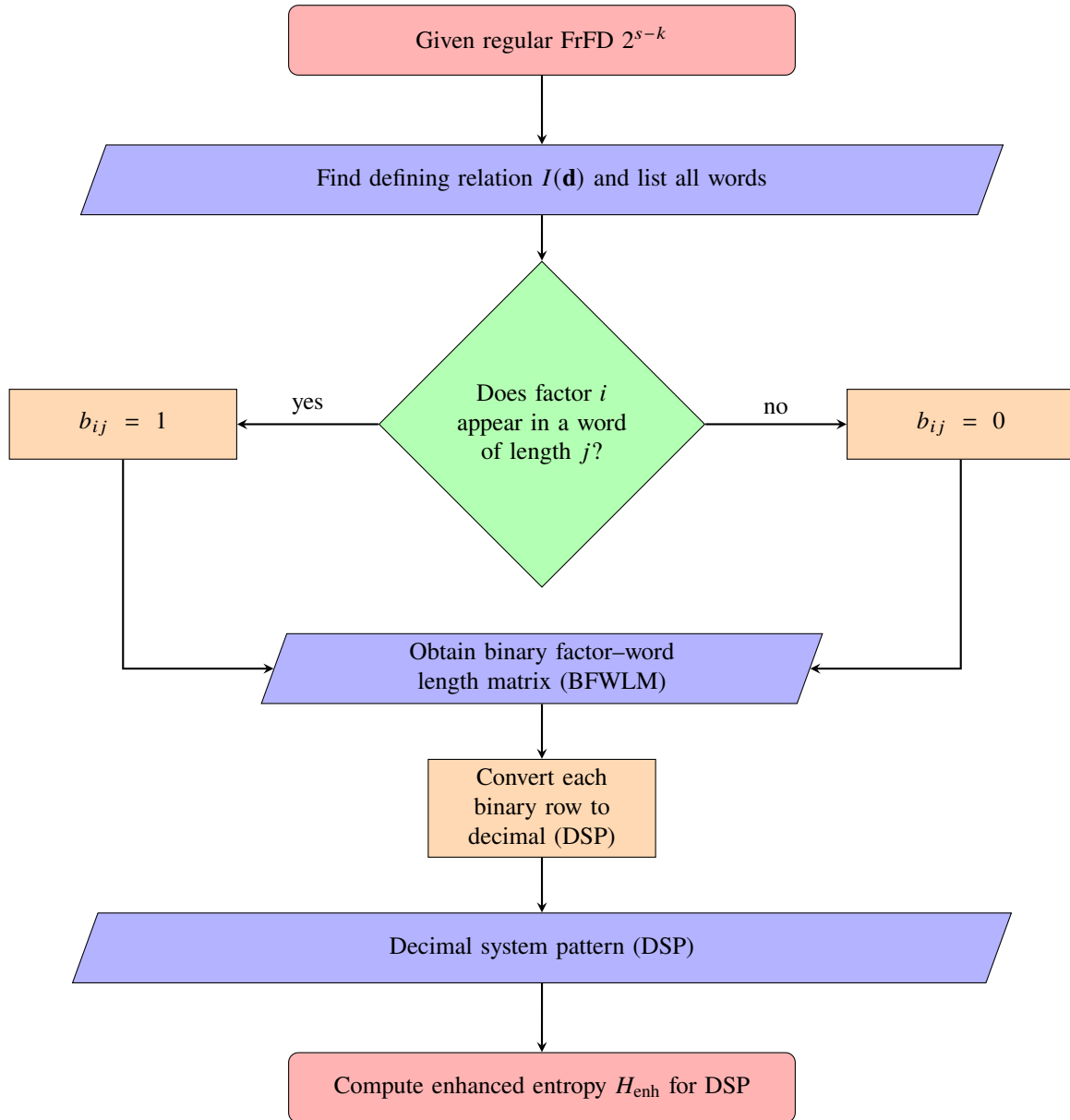


Figure 3: Flowchart for Method 3 (EE–DSP–BFWLM) for regular FrFDs.

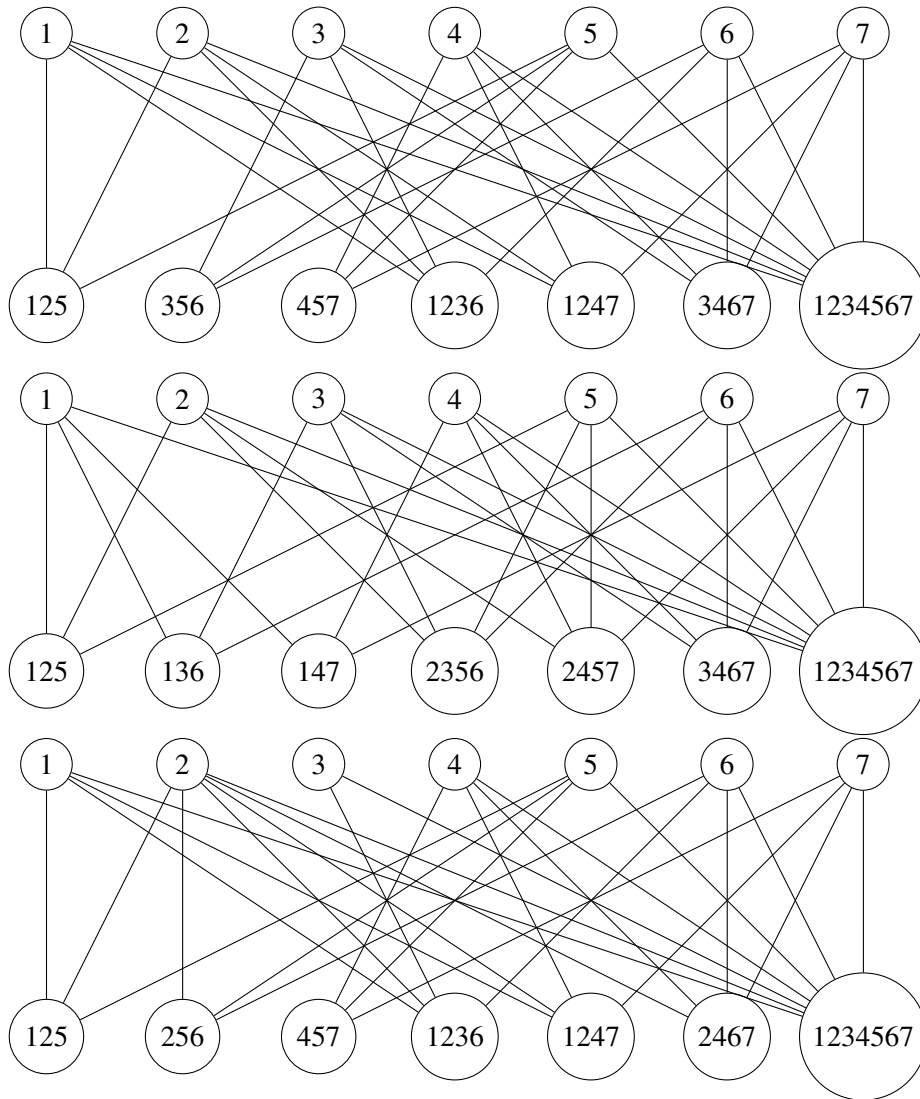


Figure 4: Maps of the relationships among the factors of  $d_1$ ,  $d_2$  and  $d_3$ , respectively.

Table 1: BFWM, DSP and  $H_{\text{enh}}$  for  $\mathbf{d}_1$ ,  $\mathbf{d}_2$  and  $\mathbf{d}_3$  (Algorithm 1).

$\mathbf{d}_1$	125	356	457	1236	1247	3467	1234567	DSP
1	1	0	0	1	1	0	1	77
2	1	0	0	1	1	0	1	77
3	0	1	0	1	0	1	1	43
4	0	0	1	0	1	1	1	23
5	1	1	1	0	0	0	1	113
6	0	1	0	1	0	1	1	43
7	0	0	1	0	1	1	1	23
$H_{\text{enh}}(\text{DSP}(\mathbf{d}_1)) = 6522.9615$								
$\mathbf{d}_2$	125	136	147	2356	2457	3467	1234567	DSP
1	1	1	1	0	0	0	1	113
2	1	0	0	1	1	0	1	77
3	0	1	0	1	0	1	1	43
4	0	0	1	0	1	1	1	23
5	1	0	0	1	1	0	1	77
6	0	1	0	1	0	1	1	43
7	0	0	1	0	1	1	1	23
$H_{\text{enh}}(\text{DSP}(\mathbf{d}_2)) = 6522.9615$								
$\mathbf{d}_3$	125	256	457	1236	1247	2467	1234567	DSP
1	1	0	0	1	1	0	1	77
2	1	1	0	1	1	1	1	111
3	0	0	0	1	0	0	1	9
4	0	0	1	0	1	1	1	23
5	1	1	1	0	0	0	1	113
6	0	1	0	1	0	1	1	43
7	0	0	1	0	1	1	1	23
$H_{\text{enh}}(\text{DSP}(\mathbf{d}_3)) = 9348.7452$								

Table 2: LWLM and  $H_{\text{enh}}$  for  $\mathbf{d}_1$ ,  $\mathbf{d}_2$  and  $\mathbf{d}_3$  (Algorithm 2).

Factor	Word length						
	1	2	3	4	5	6	7
1	0	0	1	2	0	0	1
2	0	0	1	2	0	0	1
3	0	0	1	2	0	0	1
4	0	0	1	2	0	0	1
5	0	0	3	0	0	0	1
6	0	0	1	2	0	0	1
7	0	0	1	2	0	0	1
$H_{\text{enh}}(\mathbf{d}_1) = 2.0955$							
Factor	1	2	3	4	5	6	7
1	0	0	3	0	0	0	1
2	0	0	1	2	0	0	1
3	0	0	1	2	0	0	1
4	0	0	1	2	0	0	1
5	0	0	1	2	0	0	1
6	0	0	1	2	0	0	1
7	0	0	1	2	0	0	1
$H_{\text{enh}}(\mathbf{d}_2) = 2.0955$							
Factor	1	2	3	4	5	6	7
1	0	0	1	2	0	0	1
2	0	0	2	3	0	0	1
3	0	0	0	1	0	0	1
4	0	0	1	2	0	0	1
5	0	0	3	0	0	0	1
6	0	0	1	2	0	0	1
7	0	0	1	2	0	0	1
$H_{\text{enh}}(\mathbf{d}_3) = 2.4512$							

Table 3: BFWLM, DSP and  $H_{\text{enh}}$  for  $\mathbf{d}_1$ ,  $\mathbf{d}_2$  and  $\mathbf{d}_3$  (Algorithm 3).

Factor	Word length							DSP
	1	2	3	4	5	6	7	
1	0	0	1	1	0	0	1	25
2	0	0	1	1	0	0	1	25
3	0	0	1	1	0	0	1	25
4	0	0	1	1	0	0	1	25
5	0	0	1	0	0	0	1	17
6	0	0	1	1	0	0	1	25
7	0	0	1	1	0	0	1	25
$H_{\text{enh}}(\text{DSP}(\mathbf{d}_1)) = 162.919$								
Factor	1	2	3	4	5	6	7	DSP
1	0	0	1	0	0	0	1	17
2	0	0	1	1	0	0	1	25
3	0	0	1	1	0	0	1	25
4	0	0	1	1	0	0	1	25
5	0	0	1	1	0	0	1	25
6	0	0	1	1	0	0	1	25
7	0	0	1	1	0	0	1	25
$H_{\text{enh}}(\text{DSP}(\mathbf{d}_2)) = 162.919$								
Factor	1	2	3	4	5	6	7	DSP
1	0	0	1	1	0	0	1	25
2	0	0	1	1	0	0	1	25
3	0	0	0	1	0	0	1	9
4	0	0	1	1	0	0	1	25
5	0	0	1	0	0	0	1	17
6	0	0	1	1	0	0	1	25
7	0	0	1	1	0	0	1	25
$H_{\text{enh}}(\text{DSP}(\mathbf{d}_3)) = 253.066$								

### 4 The Proposed Methods for Detecting Non-Isomorphic Non-Regular FrFDs

Unlike regular designs, non-regular orthogonal arrays do not possess a defining relation. Therefore, the binary factor–word matrices introduced in Section 3 cannot be constructed. Instead, we exploit the Hamming distance between runs to capture structural information. The overall workflow is illustrated in Figure 5.

For a design  $\mathbf{d}$  of size  $n \times m$ , we compute the Hamming distance between every pair of distinct rows (as defined in Section 2). This yields a binary matrix of size  $\binom{n}{2} \times m$ : each row corresponds to a pair of runs, and each column indicates whether the two runs differ in that factor. The matrix is binary because we only record agreement/disagreement, not the magnitude of the difference. This transformation discards the actual level values but preserves the pattern across runs, which is sufficient for distinguishing non-isomorphic designs.

Each binary row is then interpreted as a binary number and converted to a decimal integer:

$$\text{Decimal}(r) = \sum_{k=1}^m b_{rk} \cdot 2^{k-1},$$

where  $b_{rk}$  is the entry in the  $k$ -th column of the  $r$ -th row. The resulting vector  $\mathbf{V}$  of length  $\binom{n}{2}$  is the

decimal signature of the design. Algorithm 1 summarises the computation.

Once the decimal vector  $\mathbf{V}$  is obtained, we compute the frequencies of each distinct value and the corresponding probabilities  $p_i = \text{frequency}_i / \binom{n}{2}$ . Following the definition in Section 3, we use the enhanced entropy

$$H_{\text{enh}} = - \sum_i p_i \log p_i \cdot X_i^2,$$

where  $X_i$  are the distinct decimal values. The square weighting emphasises larger decimal numbers, which often correspond to more complex run-pair patterns. For comparison, we also consider linear weighting ( $H_{\text{new}} = - \sum p_i \log p_i \cdot X_i$ ) and square-root weighting ( $H_{\text{sqr}} = - \sum p_i \log p_i \cdot \sqrt{X_i}$ ). All three weighted versions are more discriminative than the classical entropy, with the squared version ( $H_{\text{enh}}$ ) providing the clearest separation, as shown below.

Table 5 reports these entropy measures for the three non-isomorphic  $\mathcal{L}_{18}(3^7)$  designs in Table 4. While the classical entropies differ only slightly, the weighted entropies—especially  $H_{\text{enh}}$ —exhibit large gaps, making the designs perfectly distinguishable.

For small collections of designs, a simpler method is to directly compare the sorted decimal vectors  $\mathbf{V}$ . Two designs are isomorphic if their sorted vectors are identical. The number of differing elements after matching common values quantifies the degree of non-isomorphism. Algorithm 2 performs this comparison, and Table 6 shows the pairwise differences for the three  $\mathcal{L}_{18}(3^7)$  designs.

#### 4.1 The Performance of the Proposed Method

To further validate the method, we applied it to three well-known families of non-regular non-isomorphic designs:

- $\mathcal{L}_{18}(3^7)$ : 18 runs, 7 factors, three levels (3 non-isomorphic designs).
- $\mathcal{L}_{32}(4^9)$ : 32 runs, 9 factors, four levels (20 non-isomorphic designs).
- $\mathcal{L}_{27}(3^{13})$ : 27 runs, 13 factors, three levels (68 non-isomorphic designs).

For each design, we computed the Hamming distance matrix, converted it to a decimal vector, and then calculated  $H_{\text{enh}}$ . The results are summarised in Tables 5, 7, and 8. In every case, all non-isomorphic designs received distinct  $H_{\text{enh}}$  values, achieving perfect discrimination. From these tables, we can conclude that:

- $\mathcal{L}_{18}(3^7)$  (Table 5): The three designs yield  $H_{\text{enh}} = 7442.7389, 7727.8577, \text{ and } 6507.1469$ , clearly separated.
- $\mathcal{L}_{32}(4^9)$  (Table 7): The 20 designs also exhibit distinct  $H_{\text{enh}}$  values, confirming the method's applicability to higher-level designs.
- $\mathcal{L}_{27}(3^{13})$  (Table 8): All 68 designs have distinct  $H_{\text{enh}}$  values, ranging from  $13.4 \times 10^6$  to  $135.5 \times 10^6$ . This is a substantial improvement over classical criteria such as GWLP and HDP, which fail to separate many of these designs.

The proposed method therefore provides a unified, simple, and powerful tool for isomorphism detection in both regular and non-regular orthogonal designs. Its computational cost is low: it requires only the construction of a binary matrix (factor–word for regular designs, Hamming distance for non-regular) and an entropy calculation.

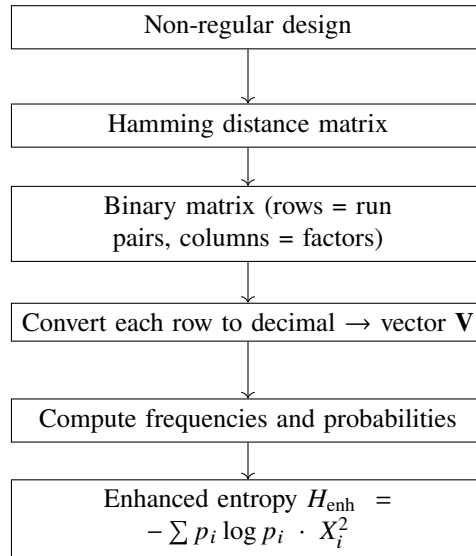


Figure 5: Workflow for detecting non-isomorphic non-regular designs.

**Algorithm 1** Hamming distance matrix**Input:** Design matrix  $\mathbf{d}$  of size  $n \times m$ **Output:** Hamming distance matrix  $X$  of size  $\binom{n}{2} \times m$ 

```

1  $c \leftarrow 1$  for  $i = 1$  to  $n$  do
2   for  $j = i + 1$  to  $n$  do
3     for  $k = 1$  to  $m$  do
4       if  $\mathbf{d}_{ik} = \mathbf{d}_{jk}$  then
5          $X_{c,k} \leftarrow 1$ 
6       end
7       else
8          $X_{c,k} \leftarrow 0$ 
9       end
10    end
11     $c \leftarrow c + 1$ 
12  end
13 end
14 return  $X$ 

```

Table 4: The three non-isomorphic orthogonal designs with 18 runs, three-levels and 7 factors  $\mathcal{L}_{18}(3^7)$ .

$L_{18}(3^7)$	$L_{18}(3^7)$	$L_{18}(3^7)$
0 0 0 0 0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0
0 1 1 1 1 0 0	0 1 1 1 1 0 0	0 1 1 1 1 1 0
0 2 1 1 2 1 1	0 2 1 1 2 1 1	0 2 1 1 0 2 1
0 0 2 2 1 1 2	0 0 2 2 1 1 2	0 0 2 2 1 2 2
0 1 2 2 0 2 1	0 1 2 2 0 2 1	0 1 2 2 2 0 1
0 2 0 0 2 2 2	0 2 0 0 2 2 2	0 2 0 0 2 1 2
1 0 1 0 1 2 1	1 0 1 0 1 2 1	1 0 1 0 1 0 1
1 1 1 0 0 1 2	1 1 1 0 0 1 2	1 1 1 0 2 2 2
1 2 2 1 0 2 0	1 2 2 1 0 2 0	1 2 2 1 0 0 2
1 0 2 1 2 0 2	1 0 2 1 2 0 2	1 0 2 1 2 1 0
1 1 0 2 2 1 0	1 1 0 2 2 0 1	1 1 0 2 0 1 1
1 2 0 2 1 0 1	1 2 0 2 1 1 0	1 2 0 2 1 2 0
2 0 1 2 2 2 0	2 0 1 2 2 2 0	2 0 1 2 0 1 2
2 1 2 0 2 0 1	2 1 2 0 2 1 0	2 1 2 0 0 2 0
2 2 1 2 0 0 2	2 2 1 2 0 0 2	2 2 1 2 2 0 0
2 0 0 1 0 1 1	2 0 0 1 0 1 1	2 0 0 1 2 2 1
2 1 0 1 1 2 2	2 1 0 1 1 2 2	2 1 0 1 1 0 2
2 2 2 0 1 1 0	2 2 2 0 1 0 1	2 2 2 0 1 1 1

Table 5: The Unique DSP for the BHDM, the corresponding frequency and probability, and the resulting EE for the three non-isomorphic orthogonal designs with 18 runs, three-levels and 7 factors  $\mathcal{L}_{18}(3^7)$ .

$L_{18}(3)$	$L_{18}(3)$	$L_{18}(3)$
<b>Unique DSP:</b> 3 5 6 8 9 10 11 12 13 16 17 18 19 20 21 22 33 34 36 40 42 44 48 49 50 52 64 65 66 67 68 69 88 96 98 100	<b>Unique DSP:</b> 3 5 6 8 9 10 11 12 13 14 16 17 18 19 20 21 22 33 34 36 40 41 42 44 48 49 50 52 64 65 66 67 68 69 70 88 96 97 98 100	<b>Unique DSP:</b> 3 5 6 8 11 13 14 17 18 20 33 34 36 41 42 44 48 65 66 68 88 96
<b>Frequencies:</b> 6 6 6 4 7 7 1 7 1 10 4 4 1 4 1 3 6 6 6 7 1 1 4 3 1 1 4 7 7 1 7 1 9 7 1 1	<b>Frequencies:</b> 6 6 6 6 6 6 1 6 1 6 6 6 1 6 1 6 6 6 6 1 1 1 6 1 1 1 6 6 6 1 6 1 9 6 1 1 1	<b>Frequencies:</b> 6 6 6 18 3 3 3 9 9 9 6 6 6 3 3 3 9 9 9 9 9 9
<b>Probabilities:</b> 0.0392 0.0392 0.0392 0.0261 0.0458 0.0458 0.0065 0.0458 0.0065 0.0654 0.0261 0.0261 0.0065 0.0261 0.0065 0.0196 0.0392 0.0392 0.0392 0.0458 0.0065 0.0065 0.0261 0.0196 0.0065 0.0065 0.0261 0.0458 0.0458 0.0065 0.0458 0.0065 0.0588 0.0458 0.0065 0.0065	<b>Probabilities:</b> 0.0392 0.0392 0.0392 0.0392 0.0392 0.0392 0.0065 0.0392 0.0065 0.0065 0.0392 0.0065 0.0065 0.0065 0.0392 0.0065 0.0065 0.0392 0.0392 0.0065 0.0392 0.0065 0.0065 0.0588 0.0392 0.0065 0.0065	<b>Probabilities:</b> 0.0392 0.0392 0.0392 0.1176 0.0196 0.0196 0.0196 0.0588 0.0588 0.0588 0.0392 0.0392 0.0392 0.0196 0.0196 0.0196 0.0588 0.0588 0.0588 0.0588 0.0588
<b>EE</b> = 7442.7389	<b>EE</b> = 7727.8577	<b>EE</b> = 6507.1469

---

**Algorithm 2** One-by-one vector comparison

---

**Input:** Two decimal vectors  $\mathbf{V}_1, \mathbf{V}_2$

**Output:** Number of unmatched elements in  $\mathbf{V}_2$

```

temp ← V2 for each q in V1 do
    if q is in temp then
        | remove one occurrence of q from temp
    end
end
return length(temp)
    
```

---

Table 6: Pairwise differences of decimal vectors for  $\mathcal{L}_{18}(3^7)$  designs.

Pair	Difference
$L_{18}^1$ vs $L_{18}^2$	16
$L_{18}^2$ vs $L_{18}^3$	48
$L_{18}^1$ vs $L_{18}^3$	56
Total	153

Table 7: The EE for the 20 non-isomorphic orthogonal designs with 32 runs, four-levels and 9 factors  $\mathcal{L}_{32}(4^9)$ .

Design	EE (DSP(BHDM))	Design	EE (DSP(BHDM))
$L_{32}^1$	87002.5028	$L_{32}^2$	100029.9907
$L_{32}^3$	98237.4543	$L_{32}^4$	101876.1627
$L_{32}^5$	90243.3392	$L_{32}^6$	99516.6647
$L_{32}^7$	101401.8044	$L_{32}^8$	103218.0371
$L_{32}^9$	98736.2017	$L_{32}^{10}$	98411.0713
$L_{32}^{11}$	95400.7960	$L_{32}^{12}$	88695.4173
$L_{32}^{13}$	91460.1547	$L_{32}^{14}$	90635.8015
$L_{32}^{15}$	90635.1754	$L_{32}^{16}$	80034.8911
$L_{32}^{17}$	87002.5028	$L_{32}^{18}$	87002.5028
$L_{32}^{19}$	99608.0704	$L_{32}^{20}$	106612.5869

Table 8: The EE for the 68 non-isomorphic orthogonal designs with 27 runs, three-levels and 13 factors  $\mathcal{L}_{27}(3^{13})$ .

Design	EE (DSP(BHDM))	Design	EE (DSP(BHDM))
$L_{27}^1$	26478434.6972	$L_{27}^2$	28210633.1719
$L_{27}^3$	35461747.7029	$L_{27}^4$	34875721.1992
$L_{27}^5$	34091803.3395	$L_{27}^6$	29942831.6466
$L_{27}^7$	29365432.1551	$L_{27}^8$	42660466.7021
$L_{27}^9$	33992106.8486	$L_{27}^{10}$	35658109.3546
$L_{27}^{11}$	34399118.8812	$L_{27}^{12}$	35846460.0103
$L_{27}^{13}$	35914764.0624	$L_{27}^{14}$	46866646.0980
$L_{27}^{15}$	49701178.1919	$L_{27}^{16}$	29584301.3117
$L_{27}^{17}$	37073765.6432	$L_{27}^{18}$	52113742.9587
$L_{27}^{19}$	36778766.9821	$L_{27}^{20}$	36271749.7139
$L_{27}^{21}$	36651251.4897	$L_{27}^{22}$	36529977.0449
$L_{27}^{23}$	36356536.1011	$L_{27}^{24}$	36822641.5273
$L_{27}^{25}$	43134035.0550	$L_{27}^{26}$	36640174.3614
$L_{27}^{27}$	45489579.1903	$L_{27}^{28}$	37339839.3755
$L_{27}^{29}$	36576276.6457	$L_{27}^{30}$	37207067.9487
$L_{27}^{31}$	37207067.9487	$L_{27}^{32}$	40337964.0824
$L_{27}^{33}$	40464331.2155	$L_{27}^{34}$	44231292.0166
$L_{27}^{35}$	43786315.1577	$L_{27}^{36}$	44244472.5400
$L_{27}^{37}$	37140683.9344	$L_{27}^{38}$	36530100.0341
$L_{27}^{39}$	37028545.9749	$L_{27}^{40}$	43473648.0992
$L_{27}^{41}$	43473648.0992	$L_{27}^{42}$	43245972.6665
$L_{27}^{43}$	40246942.2972	$L_{27}^{44}$	40360694.1816
$L_{27}^{45}$	44689887.0260	$L_{27}^{46}$	43142060.3177
$L_{27}^{47}$	42501220.4036	$L_{27}^{48}$	42548689.0337
$L_{27}^{49}$	42548689.0337	$L_{27}^{50}$	40347323.9459
$L_{27}^{51}$	37775969.9868	$L_{27}^{52}$	38289694.0159
$L_{27}^{53}$	135523127.8624	$L_{27}^{54}$	38149617.4039
$L_{27}^{55}$	38807333.8702	$L_{27}^{56}$	43899235.3840
$L_{27}^{57}$	42449698.2238	$L_{27}^{58}$	40647118.3014
$L_{27}^{59}$	40550371.4039	$L_{27}^{60}$	46223788.6113
$L_{27}^{61}$	43430122.8484	$L_{27}^{62}$	45916386.9060

Continued on next page

Table 8 – continued

Design	EE (DSP(BHDM))	Design	EE (DSP(BHDM))
$L_{27}^{63}$	45004852.7863	$L_{27}^{64}$	46327503.9580
$L_{27}^{65}$	43873980.7261	$L_{27}^{66}$	49648378.8438
$L_{27}^{67}$	40470620.5025	$L_{27}^{68}$	48297996.0132

## 5 Conclusion and Future Work

In this paper, we have developed a unified hybrid framework for detecting non-isomorphic orthogonal designs, applicable to both regular and non-regular fractional factorial designs. The proposed approach integrates three key components: (i) a decimal-system transformation that converts binary structural information into a compact numerical vector; (ii) an enhanced entropy measure that incorporates the magnitude of the decimal values via a squared weighting; and (iii) a direct vector comparison method for small-scale verification.

For regular designs, we introduced three complementary algorithms—EE-DSP-BFWM, EE-LWLM-LWLM, and EE-DSP-BFWLM—each exploiting the defining relations to construct binary factor–word or factor–word-length matrices. For non-regular designs, we replaced the defining relation with a Hamming-distance matrix between runs, then applied the same decimal-entropy procedure. Extensive experiments on three challenging families— $\mathcal{L}_{18}(3^7)$  (3 non-isomorphic classes),  $\mathcal{L}_{27}(3^{13})$  (68 non-isomorphic classes), and  $\mathcal{L}_{32}(4^9)$  (20 non-isomorphic classes)—demonstrated that the proposed method achieves *perfect discrimination*, i.e., every non-isomorphic design receives a distinct enhanced entropy value. This is a substantial improvement over classical criteria such as GWLP and HDP, which fail to separate many of these designs, and over projection-based methods that are computationally expensive and may still leave ambiguities.

The main advantages of our method are: *Simplicity*: It requires only elementary binary-to-decimal conversion and entropy calculation, avoiding complex iterative searches or high-dimensional projections. *Universality*: The same core idea works for regular and non-regular designs. *Computational efficiency*: The time complexity is dominated by the construction of the binary matrix (factor–word for regular designs, Hamming distance for non-regular designs) and the entropy computation, both of which scale polynomially with the number of runs and factors. *Interpretability*: The enhanced entropy provides a single scalar summary that reflects both the probability distribution and the magnitudes of the decimal values, making it easy to compare designs.

Several directions are worth exploring: *Integration into design search algorithms*. The proposed isomorphism detection can be embedded into computer-based optimal design generators (e.g., algorithms that search for D-optimal or uniform designs) to prune isomorphic copies and thus accelerate the search. A fast, incremental version of the method would be particularly valuable. *Theoretical analysis of the enhanced entropy*. While the empirical results show that the squared-weighted entropy perfectly separates all tested non-isomorphic families, a theoretical justification (e.g., proving that the mapping from design to decimal vector is injective up to isomorphism for certain classes) would be a significant contribution.

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